TOP SECRET [redacted] RUFF
[redacted]

[redacted]
Copy No [redacted]

SUBTASK REPORT

PROBLEMS AND POSSIBILITIES OF HIGH ALTITUDE WORLDWIDE MAPPING

Development of Mathematical System Geometry Models

Task 1, Subtask C

Prepared by Itek Data Analysis Center

Approved by [redacted]

10 December 65

Itek Corporation

HANDLE VIA
TALENT-KEYHOLE
CONTROL SYSTEM ONLY

TOP SECRET [redacted] RUFF

CONTENTS

FIGURES

TABLE

STATUS OF TASK

This report details the philosophy mathematical exercises required in the generation of the MSGM. The actual Error Model Program has been written as of the publishing of this report; however, it has not been completely debugged. It has been deemed necessary to have realistic data inputs for the program from actual operational materials; thus, the actual massaging of the program will not be completed until the first PG Mission has been realized. It is conceivable that the computer program will receive its final polishing in the actual analysis of the PG System output.

Furthermore, it is intended to incorporate both the analytical philosophy and mathematical techniques in the actual data reduction and mapping operations. This effort is to be carried forth in the continuation of this subtask and in Task 1 Subtask D, and Equipment Procedures for Calibrated Panoramic Data Handling.

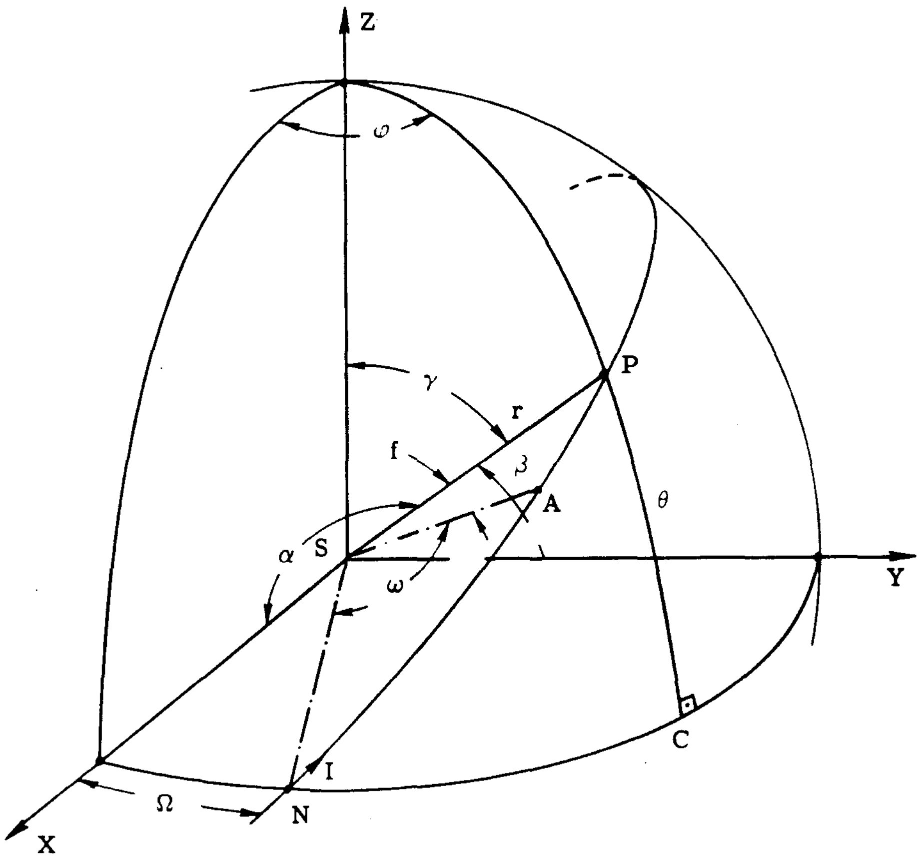


Fig. 1-1 — Orbital elements

INTRODUCTION

The objective of this report is to devise an appropriate error model, whereby the analysis of a satellite borne camera system might be facilitated.

The camera system consists of three prime sensors, viz., a nominally vertical frame camera and two inclined panoramic cameras. The latter sweep cross track, and are configured in such a way that symmetric convergent stereo coverage is furnished. This coverage is common with a portion of the stereo-overlap that is obtained with successive overlapping frame exposures.

The selected error model will consider four exposures; two panoramic and two frame, which are constrained to orbital and auxiliary data. This is sufficiently general to consider all combinations of frame and panoramic exposures, subject to constraints, with or without stereo overlap, and may be readily extended to complete blocks and strips of photographs.

The effects of various errors in the parameters of this four photoblock will be obtained through an analytical experimental design, that parallels the statistician's controlled experiments. For this design exact fictitious data is generated, and the effects of varying the accuracies of different parameters (individually and in combination) on the end result are obtained by variance-covariance analyses.

It has been suggested that the reduction of the panoramic photography might be accomplished with greater expediency and precision by using an empirical fitting process. This suggestion is based on the undefined internal geometry of many panoramic cameras and the instability of this internal geometry. Without a precise knowledge of the interior orientation, an analytical solution must fail.

Consequently this report will consist of discussions around the following topics:

1. Selection of an orbital model
2. Photogrammetric considerations
3. A generalized least squares solution
4. An empirical fitting of the panoramic material
5. The experimental design

Although the mathematical development is simple, some rather extensive algebraic expressions result. Consequently, the step by step descriptions of the fictitious model data computations, and the specific algebraic expressions for various terms, are contained as appendices to the main text.

[redacted]

τ = Epoch, the time at which the body passes through the pericenter
t = the time at which the body is at some point P
μ = Universal gravitational constant
n = mean motion of body: $n = \mu^{1/2} a^{-3/2}$
M = mean anomaly: $M = n(t - \tau)$
E = eccentric anomaly, defined by $M = E - e \sin E$
Φ = celestial longitude of body, angle XZP
θ = celestial latitude of body, angle CSP

1.3 RECTANGULAR COORDINATES XYZ OF BODY

The coordinates of P with respect to the inertial system are given by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r \begin{bmatrix} \cos \alpha \\ \cos \beta \\ \cos \gamma \end{bmatrix} = r \begin{bmatrix} \cos \theta \cos \Phi \\ \cos \theta \sin \Phi \\ \sin \theta \end{bmatrix} \tag{1}$$

where α, β, γ, are the direction angles of the vector $\vec{r}$. The direction cosines may be derived from the triangles PNX and PNY (Figure 1-1) as

$$\cos \alpha = \cos \Omega \cos (\omega + f) - \sin \Omega \sin (\omega + f) \cos I \tag{2}$$

$$\cos \beta = \sin \Omega \cos (\omega + f) + \cos \Omega \sin (\omega + f) \cos I \tag{3}$$

and

$$\cos \gamma = \sin (\omega + f) \sin I \tag{4}$$

Substituting (2), (3), and (4) into (1), expanding $\cos (\omega + f)$ and $\sin (\omega + f)$, and using the identities

$$r \cos f = a(\cos E - e) \tag{5}$$

and

$$r \sin f = a(1 - e^2)^{1/2} \sin E \tag{6}$$

reduces (1) to the form

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} A_x & B_x \\ A_y & B_y \\ A_z & B_z \end{bmatrix} \begin{bmatrix} \cos (E) - e \\ \sin E \end{bmatrix} \tag{7}$$

HANDLE VIA
TALENT-KEYHOLE
CONTROL SYSTEM ONLY

[redacted]

in which the matrix $\begin{bmatrix} A_x & B_x \\ A_y & B_y \\ A_z & B_z \end{bmatrix}$, is given by

$$\begin{bmatrix} A_x & B_x \\ A_y & B_y \\ A_z & B_z \end{bmatrix} = a \begin{bmatrix} \cos \Omega & -\cos I \sin \Omega \\ \sin \Omega & \cos I \cos \Omega \\ 0 & \sin I \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1 - e^2)^{1/2} \end{bmatrix} \qquad (8)$$

This may be expanded to yield

$$A_x = a\,(\cos \Omega \cos \omega - \sin \Omega \sin \omega \cos I) \qquad (9)$$

$$A_y = a\,(\sin \Omega \cos \omega + \cos \Omega \sin \omega \cos I) \qquad (10)$$

$$A_z = a\,(\sin I \sin \omega) \qquad (11)$$

$$B_x = -a\,(1 - e^2)^{1/2}\,(\cos \Omega \sin \omega + \sin \Omega \cos \omega \cos I) \qquad (12)$$

$$B_y = -a\,(1 - e^2)^{1/2}\,(\sin \Omega \sin \omega - \cos \Omega \cos \omega \cos I) \qquad (13)$$

$$B_z = a\,(1 - e^2)^{1/2}\,(\sin I \cos \omega) \qquad (14)$$

The positional coordinates, given by (7) are in an inertial system, with reference to which the orbital elements are given. In order that they may be referred to some other coordinate system, transformations will be necessary.

1.4 COORDINATE SYSTEMS AND TRANSFORMATIONS

For mapping purposes, a system of terrestrial rectangular coordinates, X_T Y_T Z_T is convenient, especially for points, such as vehicle positions, which are remote from the earth's surface.

This system has its origin at the center of gravity of the earth, or of the reference spheroid, and is oriented such that the $+Z_T$ axis is directed to the mean north pole,* and the mean meridian of Greenwich† lies in the X_T Z_T plane. This system is related to the geodetic longitude and latitude, λ, φ', and the geocentric longitude and latitude, λ, φ, according to

---

* As defined by Service International des Latitudes.

† As defined by Bureau International de l' Heure.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_T = \begin{bmatrix} (N + H) \cos \varphi' \cos \lambda \\ (N + H) \cos \varphi' \sin \lambda \\ [(1 - e^2) N + H] \sin \varphi' \end{bmatrix} = r \begin{bmatrix} \cos \varphi \cos \lambda \\ \cos \varphi \sin \lambda \\ \sin \varphi \end{bmatrix} \quad (15)$$

where N = radius of curvature in the prime vertical
H = elevation of point above reference datum
r = geocentric radius of point
e = eccentricity of reference ellipsoid

It is to be noted that this coordinate system is fixed with respect to the earth's surface.

For points which do not rotate with the earth as, for example, discrete orbital positions, it is desirable to define a coordinate system independent of the earth's rotation. The sidereal system is such a reference framework.

In this system, the Z_S axis corresponds with the instantaneous pole, and the X_S axis with the apparent vernal equinox, γ. The relation between the T and S systems is illustrated by Figure 1-2, the transformation being:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_S = M_1 M_2 \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_T \quad (16)$$

where

$$M_1 = \begin{bmatrix} \cos t' & -\sin t' & 0 \\ \sin t' & \cos t' & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (17)$$

and

$$M_2 = \begin{bmatrix} \cos \xi & \sin \xi \sin \eta & -\sin \xi \cos \eta \\ 0 & \cos \eta & \sin \eta \\ \sin \xi & -\cos \xi \sin \eta & \cos \xi \cos \eta \end{bmatrix} \quad (18)$$

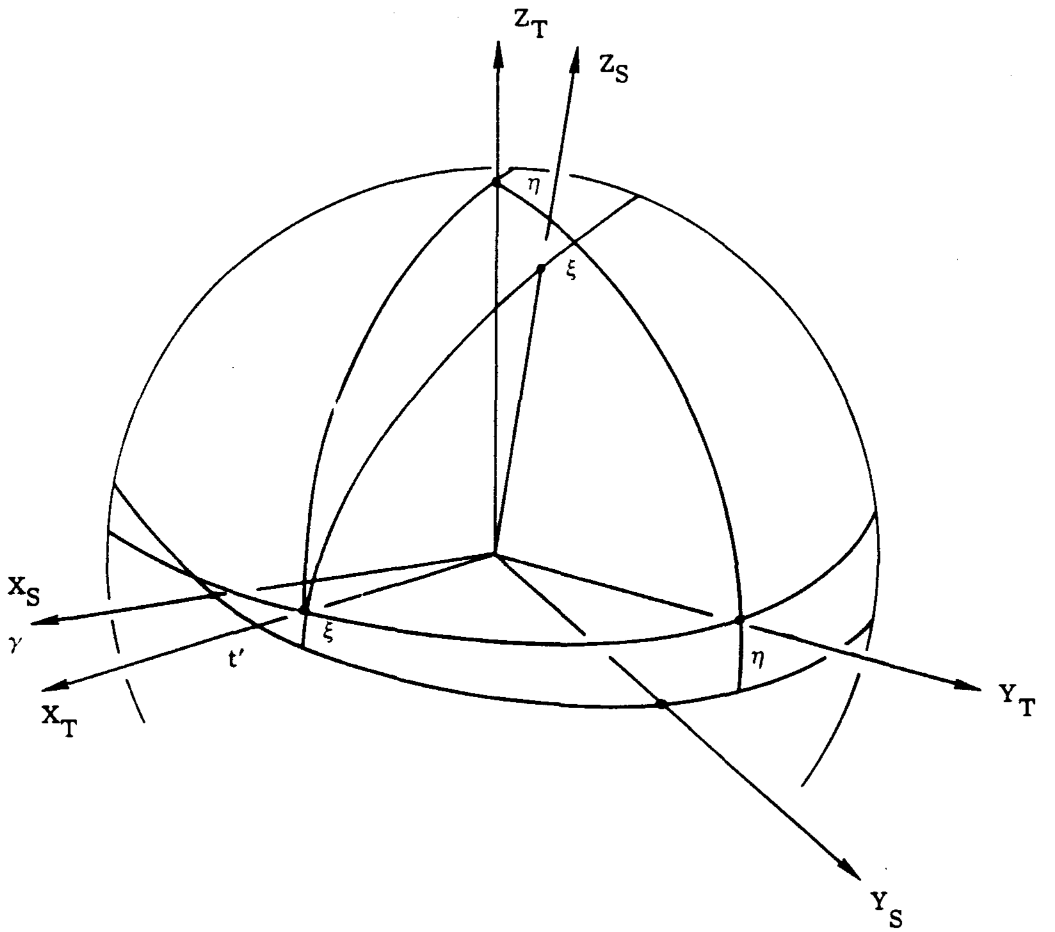


Fig. 1-2 — Siderial and terrestrial systems in which t' is the Greenwhich apparent sidereal time in UTI,* and ξ, κ are the coordinates of the instantaneous pole.†

The matrix M_2 may be considered to be a unit matrix for practical purposes. The component angles ξ, η are less than one second of arc. Furthermore, for short periods of time the changes in ξ, η are so small that Z_T and Z_S may be considered to be coincident.

---

*UTO is observed time; UTI is UTO corrected for motion of the pole; UT 2 is UTI corrected for seasonal variation in the earth's rotation. Bulletin Horaire No. 4, Series 4, 1955.

†Melchoir P. 1954. Observatoire Royale de Belgique. Monographie No. 3, Service International des Latitudes.

2. PHOTOGRAMMETRIC CONSIDERATIONS

2.1 SYSTEM CONFIGURATION

Consider two overlapping frame photographs, nominally vertical, exposed at times t_1 and t_4. Let two symmetric convengent panoramic photographs, exposed at times t_2 and t_3 overlap within the stereo model formed by the frame photographs. This is illustrated by Figure 2-1.

2.2 PROJECTIVE RELATIONSHIPS

For convenience, let the panoramic photography be transformed into equivalent frame photography, according to Figure 2-2.

Define the principal point as that position on the cylindrical panoramic format intersected by the camera z-axis when that axis is at the mid-point of the scan. The equivalent frame photograph is tangent to the generator passing through this point.

An image point x′, y′ on the pan photograph may be transformed into the equivalent frame coordinates x, y. Consider the case for diapositives, and define the +y′-axis in the flight direction, and the +z′-axis upwards. This right hand system is illustrated in Figure 2-2.

The scan angle ranges from $-\alpha$ through zero to $+\alpha$ as the optical axis scans from +x through zero to -x. Let the scan rate, $\dot{\alpha}$, be signed positive for a forward looking camera. Then

$$
\begin{aligned}
(x' - x_0') &= -f\alpha \\
(y' - y_0') &= (y - y_0)\cos\alpha \\
(x - x_0) &= -f\tan\alpha
\end{aligned}
\qquad (19)
$$

with image motion compensation given by

$$\frac{V}{(Z - Z_0)}\,\frac{-f}{\dot{\alpha}}\,\sin\alpha = C\sin\alpha$$

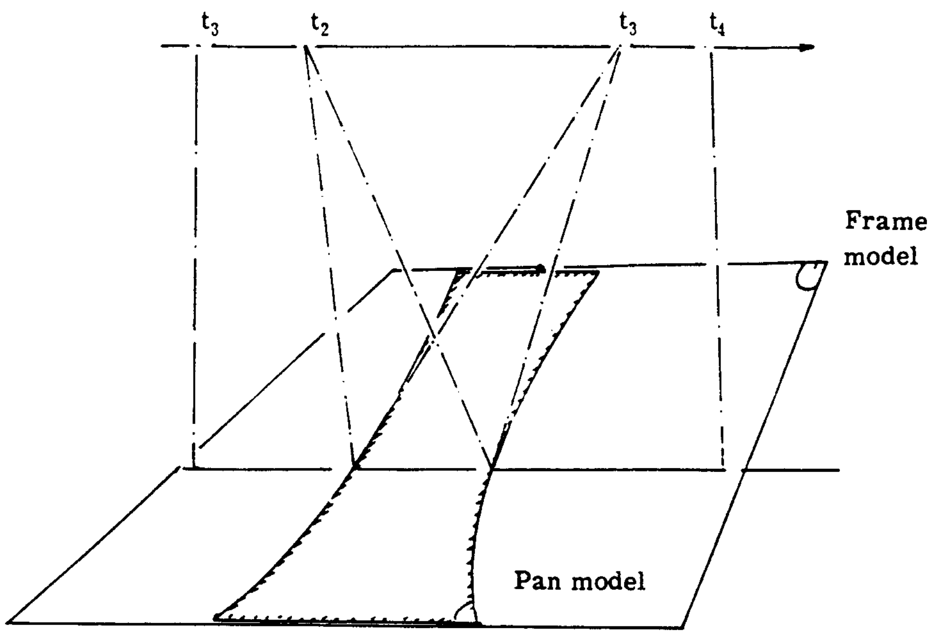


Fig. 2-1 — System configuration

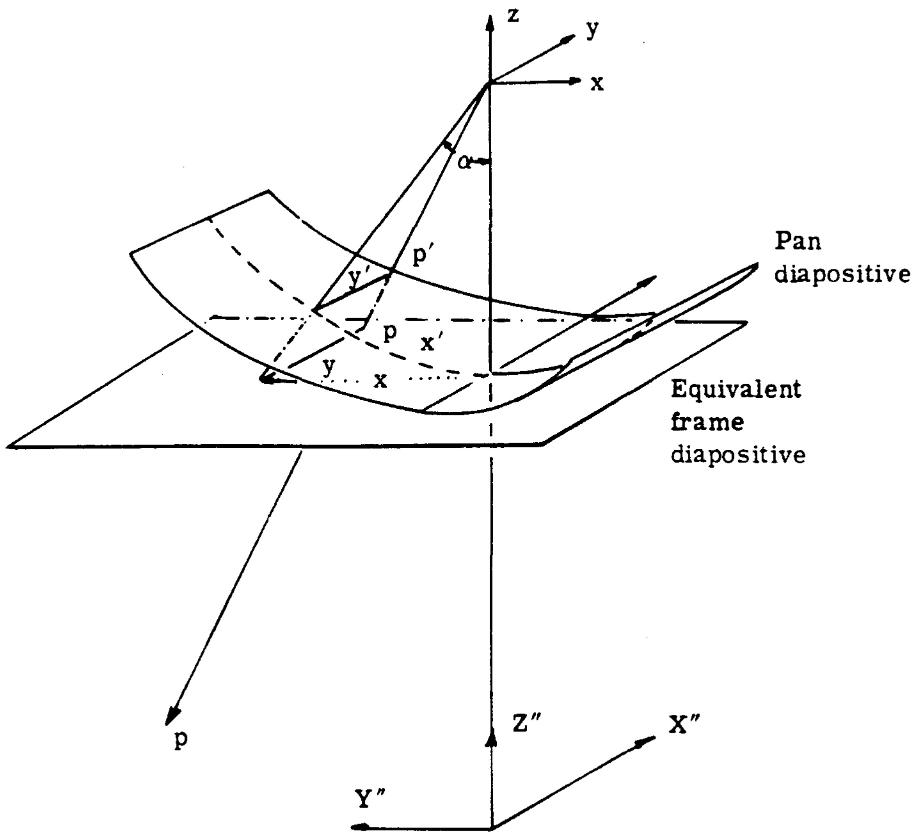


Fig. 2-2 — Pan geometry

where

$$C = \frac{Vf}{H\dot{\alpha}} \tag{20}$$

then

$$(y - y_0) = [(y' - y_0') + C \sin \alpha]/\cos \alpha$$

It is to be noted that

$$\dot{\alpha}\ (t_i - t_0) = \alpha_i = (x' - x_0')/ - f'$$

The aft looking camera scans in the opposite direction, so that $\dot{\alpha}$ is negative, and α goes from + to - values as the scan goes from $-x''$ to $+x''$. As before

$$(x'' - x_0'') = - f'' \alpha$$

$$(x - x_0) \quad = - f'' \tan \alpha$$

$$(y'' - y_0'') = (y - y_0) \cos \alpha$$

noting that α is + when x'' is -; and the image motion is given by

$$\frac{V}{(Z - Z_0)} \cdot \frac{-f''}{\dot{\alpha}} \cdot \sin \alpha = -C \sin \alpha$$

where $\dot{\alpha}$ is unsigned, or $+C \sin \alpha$ if $\dot{\alpha}$ is signed. Then

$$(y - y_0) = [(y'' - y_0'') + C \sin \alpha]/\cos \alpha$$

provided that $\dot{\alpha}$ carries the appropriate sign.

Any image vector $\begin{bmatrix} x_i - x_0 \\ y_i - y_0 \\ - f \end{bmatrix}$ in the equivalent frame coordinate system, is related to the corresponding object space vector, $\begin{bmatrix} X_i - X_0 \\ Y_i - Y_0 \\ Z_i - Z_0 \end{bmatrix}$, according to

$$\vec{x}_i = M_j \vec{X}_i \tag{21}$$

from which the projective relationships

$$x_i = x_0 - f \left[\frac{m_{11}(X_i - X_0) + m_{12}(Y_i - Y_0) + m_{13}(Z_i - Z_0)}{m_{31}(X_i - X_0) + m_{32}(Y_i - Y_0) + m_{33}(Z_i - Z_0)}\right] \tag{22}$$

and

$$y_i = y_0 - f \left[\frac{m_{21}(X_i - X_0) + m_{22}(Y_i - Y_0) + m_{23}(Z_i - Z_0)}{m_{31}(X_i - X_0) + m_{32}(Y_i - Y_0) + m_{33}(Z_i - Z_0)}\right] \tag{23}$$

are obtained.

In these expressions f is the camera focal length; m_{jk} is an element of the orientation matrix M_i; X_0, Y_0, Z_0 are the coordinates of the exposure station in the XYZ system; X_i, Y_i, Z_i, are the coordinates of the ground point corresponding to the image point x_i, y_i, f.

2.3 CAMERA ORIENTATION

In conjunction with the panoramic cameras, horizon sensors are utilized, which yield initial and final values of the camera roll and pitch with respect to the local horizon. At these specific times, the local horizon is equivalent to the tangent plane at the nadir point, which is defined as the point at which the position vector $\vec{r}$ to the air station intersects the reference ellipsoid. The longitude, λ_0, and geocentric latitude, Φ_0, of the nadir, as illustrated by Figure 2-3, are determined from

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}_S = r \begin{bmatrix} \cos\Phi_0 \cos\lambda_0 \\ \cos\Phi_0 \sin\lambda_0 \\ \sin\Phi_0 \end{bmatrix} \tag{24}$$

according to

$$\lambda_0 = \cos^{-1}\left[(X_0)(X_0^2 + Y_0^2)^{-1/2}\right], \text{ for } X_0 \geqslant Y_0 \tag{25}$$

or

$$\lambda_0 = \sin^{-1}\left[(Y_0)(X_0^2 + Y_0^2)^{-1/2}\right], \text{ for } X_0 \leqslant Y_0 \tag{26}$$

and

$$\Phi_0 = \tan^{-1}\left[(Z_0)(X_0^2 + Y_0^2)^{-1/2}\right] \tag{27}$$

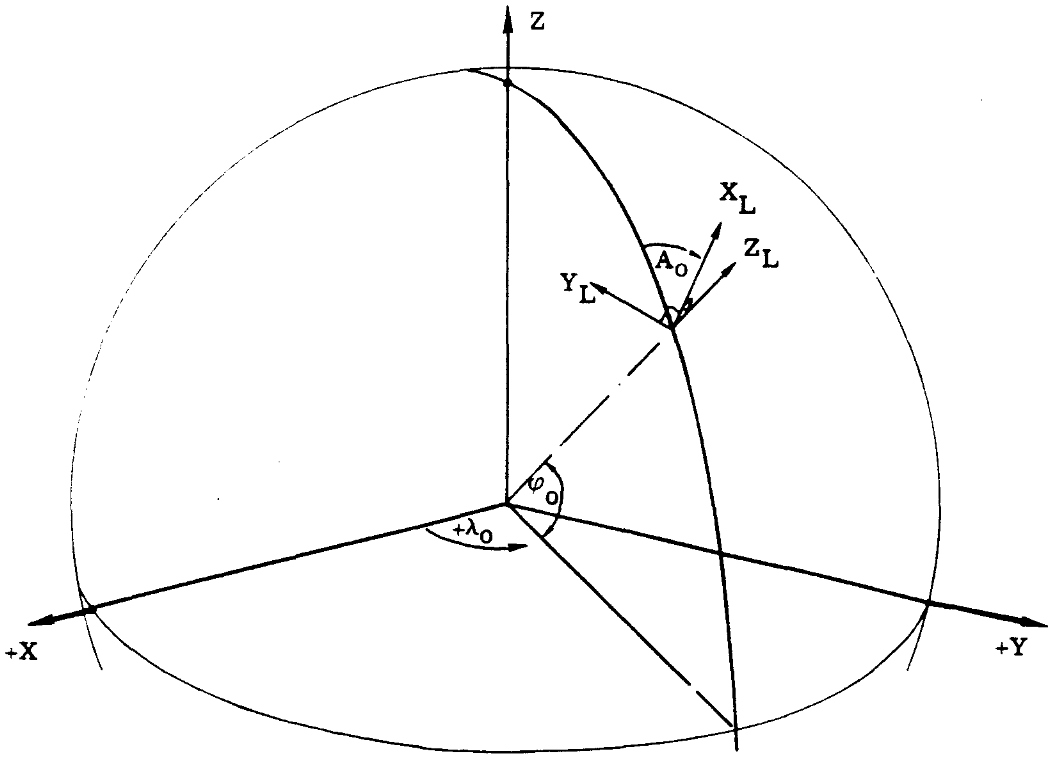


Fig. 2-3 — Geocentric and local coordinate systems

With reference to Equation (21), the orientation matrix rotating the ground coordinates into the photo system may be expressed as

$$M_j = \begin{bmatrix} \cos\kappa_j & \sin\kappa_j & 0 \\ -\sin\kappa_j & \cos\kappa_j & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi_j & 0 & -\sin\varphi_j \\ 0 & 1 & 0 \\ \sin\varphi_j & 0 & \cos\varphi_j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_j & \sin\omega_j \\ 0 & -\sin\omega_j & \cos\omega_j \end{bmatrix} \quad (28)$$

The orientation between the local coordinate system and the geocentric terrestrial system is determined directly from Figure 2-3 as

$$O_j = \begin{bmatrix} \cos A_0 & -\sin A_0 & 0 \\ \sin A_0 & \cos A_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\Phi_0 & 0 & \cos\Phi_0 \\ 0 & 1 & 0 \\ -\cos\Phi_0 & 0 & \sin\Phi_0 \end{bmatrix} \begin{bmatrix} -\cos\lambda_0 & -\sin\lambda_0 & 0 \\ \sin\lambda_0 & -\cos\lambda_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (29)$$

Equation (21) may now be rewritten as

$$\bar{x}_i = M_j O_j \left[\vec{X}_i - \vec{X}_0\right]_T \quad (30)$$

or

$$\begin{bmatrix} x_i - x_0 \\ y_i - y_0 \\ -f \end{bmatrix} = [\kappa_j][\varphi_j][\omega_j][A_0][\Phi_0][\lambda_0] \begin{bmatrix} X_i - X_0 \\ Y_i - Y_0 \\ Z_i - Z_0 \end{bmatrix} \quad (31)$$

The values of all terms in (31) pertain to a discrete time. For any specific time t_0, the corresponding terms may be evaluated and held constant. In this manner, the dynamic local system may be transformed into a static one, with reference to which subsequent computations may be performed, for any time t_i, provided that appropriate variations of $\vec{X}_0$, due to orbital motion, of ω_j, φ_j, κ_j, due to capsule tumbling and of $\vec{X}_i$ due to the earth's rotation, are applied.

This is accomplished through (16), computing in the sidereal system, and appropriately expressing variations in the angular orientations

$$\alpha_j = \omega_j,\ \varphi_j \text{ or } \kappa_j \text{ as } \alpha_j = (\alpha_0)_j + (\alpha_1 t_i)_j + (\alpha_2 t_i^2)_j + \ldots \quad (32)$$

so that the orientation matrices $[\alpha_j]$ in (31) become

$$[\alpha_j] = [\alpha_0]_j [\alpha_1 t_i]_j [\alpha_2 t_i^2]_j \quad (33)$$

although information may exist which indicates that the α_j have a specific nature, say periodic. In this event the appropriate expressions for the α_j must be used.

Denoting the product M_1M_2 in (16) by T_j, Equation (30) may be rewritten as

$$\vec{x}_i = M_jO_j \left[T_j\vec{X}_{iT} - \vec{X}_{0_S}\right] \tag{34}$$

or

$$\vec{x} = M\,\vec{X}_{iS}$$

This is in a form that is identical with (21), so that the projective Equations (22) and (23) may be used to define the functions

$$F_{ki} = G_{ki} + f\left[\frac{m_{k1}(X_i - X_0)_S + m_{k2}(Y_i - Y_0)_S + m_{k3}(Z_i - Z_0)_S}{m_{31}(X_i - X_0)_S + m_{32}(Y_i - Y_0)_S + m_{33}(Z_i - Z_0)_S}\right] \tag{35}$$

$k = 1, 2$

$G_{1j} = x_i - x_0$

$G_{2j} = y_i - y_0$

These two equations may be expressed in terms of the constituent variables ω_j, φ_j, κ_j, A_0, ... Z_0. It is noted, however that some of these are functionally related to the orbital parameters, and to each other. The constrained solution must utilize independent parameters, as outlined in the subsequent section.

3. CONSTRAINED SOLUTION

3.1 INTRODUCTION

The functions given by Equation (35) express the relationship between a series of data consisting of observed variables, and various parameters, known and unknown. Those which are known, are derived from observational data and are subject to the errors in these fundamental data.

On substituting the various parameters and data into (35), the value F_{ki} will be zero if, and only if all entries are exact. The problem is to determine the most probable values for all parameters and variables. This is accomplished through a constrained least squares solution.

3.2 LEAST SQUARES SOLUTION

Equation (35) is considered to be a function of the observed variables

$$x_i,\ y_i,\ x_i',\ y_i',\ x_i'',\ y_i''$$

where the primed and double primed coordinates refer to the two pan cameras, and a function of the parameters

$$x_0,\ y_0,\ x_0',\ y_0',\ x_0'',\ y_0'',\ f,\ f'',\ f',$$

$$\omega_j,\ \varphi_j,\ \kappa_j,\ \Omega,\ \omega,\ I,\ e,\ n,\ \tau;\ X_i,\ Y_i,\ Z_i$$

Consequently, rewrite (35) as

$$F_{ki} = f_k\,(x_i,\ y_i,\ x_i',\ y_i',\ x_i'',\ y_i'';\ x_0,\ y_0,\ f,\ x_0',\ y_0',\ f',\ x_0'',\ y_0'',\ f'';\ \omega_j,\ \varphi_j,\ \kappa_j,\ \Omega,\ \omega,\ I,\ e,\ n,\ \tau;\ X_i,\ Y_i,\ Z_i) \qquad (36)$$

Assuming that approximate values of these parameters are available, designated by a superscript °, an approximate value of Equation (36) is obtained as

$$F_{ki}^{\circ} = f_k\,[x_i^{\circ},\ y_i^{\circ},\ (x_i')^{\circ},\ (y_i')^{\circ},\ \ldots\ldots X_i^{\circ},\ Y_i^{\circ},\ Z_i^{\circ}] \qquad (37)$$

The true values of these items is obtained by applying a correction to these approximate values, according to

$$x_i = x_i^\circ + p_K^1\ \delta x_i$$
$$y_i = y_i^\circ + p_K^2\ \delta y_i$$
$$x_i' = (x')^\circ + p_K^3\ \delta x_i'$$
$$y_i' = (y')^\circ + p_K^4\ \delta y_i'$$
$$x_i'' = (x_i'')^\circ + p_K^5\ \delta x_i''$$
$$y_i'' = (y_i'')^\circ + p_K^6\ \delta y_i''$$
$$x_0 = x_0^\circ + q_K^1\ \delta x_0$$
$$y_0 = y_0^\circ + q_K^2\ \delta y_0$$
$$f = f^\circ + q_K^3\ \delta f$$
$$x_0' = (x_0')^\circ + q_K^4\ \delta x_0'$$
$$y_0' = (y'_0)^\circ + q_K^5\ \delta y_0'$$
$$f' = (f')^\circ + q_K^6\ \delta f'$$
$$x_0'' = (x_0'')^\circ + q_K^7\ \delta x_0''$$
$$y_0'' = (y_0'')^\circ + q_K^8\ \delta y_0''$$
$$f'' = (f'')^\circ + q_K^9\ \delta f''$$
$$\omega_j = \omega_j^\circ + q_K^{10}\ \delta\omega_j$$
$$\varphi_j = \varphi_j^\circ + q_K^{11}\ \delta\varphi_j$$
$$\kappa_j = \kappa_j^\circ + q_K^{12}\ \delta\eta_j$$
$$\Omega = \Omega^\circ + q_K^{13}\ \delta\Omega$$
$$\omega = \omega^\circ + q_K^{14}\ \delta\omega$$
$$I = I^\circ + q_K^{15}\ \delta I$$
$$e = e^\circ + q_K^{16}\ \delta e$$
$$n = n^\circ + q_K^{17}\ \delta n$$
$$\tau = \tau^\circ + q_K^{18}\ \delta\tau$$
$$X_i = X_i^\circ + q_K^{19}\ \delta X_i$$
$$Y_i = Y_i^\circ + q_K^{20}\ \delta Y_i$$
$$Z_i = Z_i^\circ + q_K^{21}\ \delta Z_i$$

[redacted]

the values δx_i . . . . . δZ_i being corrections, the values p,q being equal to unity or zero, depending on whether the element is unknown, or known exactly.

The reduced condition equations may be expressed in the form

$$A_X V_X + B_1 \Delta_1 + B_2 \Delta_2 + E_X = 0 \tag{38}$$

where V_X = a vector of unknown residuals
Δ_1 = a vector of unknown parameter corrections
Δ_2 = a vector of unknown parameter corrections
E_X = a vector of random variables

and

$$A_X = \frac{\partial \, (F_1, F_2)_{ij}}{\partial \, (\text{unknown variables})} = \frac{\partial \, (F_1, F_2)_{ij}}{\partial \, (x_i, y_i)_j} \tag{39}$$

$$B_1 = \frac{\partial \, (F_1, F_2)_{ij}}{\partial \, (\text{unknown parameters})} = \frac{\partial \, (F_1, F_2)_{ij}}{\partial \, (\text{all or none or } x_0, \ldots \tau)} \tag{40}$$

$$B_2 = \frac{\partial \, (F_1, F_2)_{ij}}{\partial \, (\text{parameters to be constrained})} = \frac{\partial \, (F_1, F_2)_{ij}}{\partial \, (\text{remaining parameters})} \tag{41}$$

Assume that statistical estimates of say q parameters are known, and that we wish to constrain the adjustment to fit these estimates. Designate the statistical estimates of the parameters $\vec{\beta}$ by

$$\vec{\beta}^\circ = \begin{bmatrix} \beta_1^\circ \\ \beta_2^\circ \\ \vdots \\ \beta_q^\circ \end{bmatrix} \tag{42}$$

having an associated covariance matrix σ_β:

$$\sigma_\beta = \begin{bmatrix} \sigma^2\beta_1 & \sigma\beta_1\beta_2 \ldots & \sigma\beta_1\beta_2 \\ \sigma\beta_2\beta_1 & \sigma^2\beta_2 \ \ldots & \sigma\beta_2\beta_2 \\ \vdots & \vdots & \vdots \\ \sigma\beta_2\beta_1 & \sigma\beta_q\beta_2 \ldots & \sigma^2\beta_q \end{bmatrix} \tag{43}$$

[redacted]

New linear equations may be formed and solved with (38), according to

$$\vec{\beta}^{\circ} + V^{\circ}_{\beta} = \vec{\beta}^{\circ\circ} + \vec{\Delta}_2 \qquad (44)$$

where $\vec{\beta}^{\circ\circ}$ = a current corrected value of $\vec{\beta}^{\circ}$

$\vec{V}^{\circ}_{\beta}$ = an unknown residual vector

This equation is reformed as

$$\vec{V}_{\beta} - \vec{\Delta}_2 + \vec{G} = 0 \qquad (45)$$

where $\vec{G} = \vec{\beta}^{\circ} - \vec{\beta}^{\circ\circ}$

whence Equation (38) may be rewritten as

$$A\vec{V} + B\vec{\Delta} + E = 0 \qquad (46)$$

where

$$\vec{V} = \begin{bmatrix} V_x \\ \\ V_{\beta} \end{bmatrix} \qquad (47)$$

$$\vec{\Delta} = \begin{bmatrix} \Delta_2 \\ \\ \Delta_1 \end{bmatrix} \qquad (48)$$

$$E = \begin{bmatrix} E_x \\ \\ G \end{bmatrix} \qquad (49)$$

$$A = \begin{bmatrix} A_x & 0 \\ \\ 0 & I_{qq} \end{bmatrix} \qquad (50)$$

and

$$B = \begin{bmatrix} B_2 & B_1 \\ -I_{qq} & 0 \end{bmatrix} \quad (51)$$

The conditioned solution of Equation (46) for V and Δ which minimizes

$$S = V^T \sigma^{-1} V \quad (52)$$

is required, in which

$$\sigma = \begin{bmatrix} \sigma_X & 0 \\ 0 & \sigma_\beta \end{bmatrix} \quad (53)$$

where σ_X = the covariance matrix of the observed variables

The required solution is obtained from

$$S = V^T \sigma^{-1} V - 2\lambda^T (AV + B\Delta + E) \quad (54)$$

in which λ is a vector of Lagrangian multipliers, setting

$$\frac{\delta S}{\delta V} = 0 \text{ and } \frac{\delta S}{\delta \Delta} = 0$$

to yield

$$V = \sigma A^T \lambda \quad (55)$$

and

$$-2B^T \lambda = 0 \quad (56)$$

which are combined with (46) to yield

$$\lambda = -(A \sigma A^T)^{-1} (B\Delta + E) \quad (57)$$

and

$$\Delta = -\left[B^T (A \sigma A^T)^{-1} B\right]^{-1} B^T (A \sigma A^T)^{-1} E \qquad (58)$$

This equation is evaluated by partitioning, the partitions being given by the following identities:*

$$A \sigma A^T = \begin{bmatrix} A_X \sigma_X A_X^T & 0 \\ 0 & \sigma_\beta \end{bmatrix} \qquad (59)$$

$$A \sigma A^{T^{-1}} = \begin{bmatrix} (A_X \sigma_X A_X^T)^{-1} & 0 \\ 0 & \sigma_\beta^{-1} \end{bmatrix} \qquad (60)$$

Put

$$(A_X \sigma_X A_X^T)^{-1} = W_X$$

$$B^T (A \sigma A^T)^{-1} B = \begin{bmatrix} B_2^T W_X B_2 + \sigma_\beta^{-1} & B_2^T W_X B_1 \\ B_1^T W_X B_2 & B_1^T W_X B_1 \end{bmatrix} \qquad (61)$$

and

$$B^T (A \sigma A^T)^{-1} E = \begin{bmatrix} B_2^T W_X E_X - \sigma_\beta^{-1} G \\ B_1^T W_X E_X \end{bmatrix} \qquad (62)$$

3.3 APPLICATION

In order to clarify the previous treatment, an illustrative example of applying these formulas to our problem is given.

---

*The utilization of these partitions is described in Appendix C.

For each photograph, j, one obtains for each point i

$$A_i = \begin{bmatrix} \partial F_1/\partial x_i, \partial F_1/\partial y_i, \partial F_1/\partial x_i', \partial F_1/\partial y_i', \partial F_1/\partial x_i'', \partial F_1/\partial y_i'' \\ \partial F_2/\partial x_i, \partial F_2/\partial y_i, \partial F_2/\partial x_i', \partial F_2/\partial y_i', \partial F_2/\partial x_i'', \partial F_2/\partial y_i'' \end{bmatrix} \quad (63)$$

or

$$A_i = [A_x, A_x', A_x''] \; i \quad (64)$$

For the frame camera $A_{x_i'}$, $A_{x_i''}$ are zero; for the forward looking pan camera, A_{x_i}, $A_{x_i''}$ are zero, and for the aft looking camera A_{x_i}, $A_{x_i''}$ are zero.

To each ground point that is known, a variance-covariance matrix may be assigned. For unknown ground points, coordinate values are estimated, together with large variances for these estimates.

Orbital parameters are estimated from the given ephemeris, and values of the rotations ω_j, φ_j, κ_j are determined from horizon sensors. Values for x_0, y_0, f; x_0', y_0', f′ are obtained from calibration data.

The end result is that all parameters are to be constrained—some extremely loosely—dictating that $B_1 = 0$.

Consequently, we may write (see Equation (65) on following page)

The various covariance matrices are:

σ_{x_i} for the measured frame photograph images

$\sigma_{x_i'}$ for the measured forward pan photograph images

$\sigma_{x_i''}$ for the measured rear pan photograph images

σX_i for the ground coordinates

σc for the frame camera constants

$\sigma c'$ for the forward pan camera constants

$\sigma c''$ for the aft pan camera constants

$\sigma \alpha$ for the camera tilt angles

and

σ_0 for the orbital elements

$$B_2 = \begin{bmatrix} \partial F_1/\partial x_0 & \partial F_2/\partial x_0 \\ \partial F_1/\partial y_0 & \partial F_2/\partial y_0 \\ \partial F_1/\partial f & \partial F_2/\partial f \\ \partial F_1/\partial x_0' & \partial F_2/\partial x_0' \\ \partial F_1/\partial y_0' & \partial F_2/\partial y_0' \\ \partial F_1/\partial f' & \partial F_2/\partial f' \\ \partial F_1/\partial x_0'' & \partial F_2/\partial x_0'' \\ \partial F_1/\partial y_0'' & \partial F_2/\partial y_0'' \\ \partial F_1/\partial f'' & \partial F_2/\partial f'' \\ \partial F_1/\partial \omega_j & \partial F_2/\partial \omega_j \\ \partial F_1/\partial \varphi_j & \partial F_2/\partial \varphi_j \\ \partial F_1/\partial \kappa_j & \partial F_2/\partial \kappa_j \\ \partial F_1/\partial \Omega & \partial F_2/\partial \Omega \\ \partial F_1/\partial \omega & \partial F_2/\partial \omega \\ \partial F_1/\partial I & \partial F_2/\partial I \\ \partial F_1/\partial e & \partial F_2/\partial e \\ \partial F_1/\partial n & \partial F_2/\partial n \\ \partial F_1/\partial \tau & \partial F_2/\partial \tau \\ \partial F_1/\partial X_i & \partial F_2/\partial X_i \\ \partial F_1/\partial Y_i & \partial F_2/\partial Y_i \\ \partial F_1/\partial Z_i & \partial F_2/\partial Z_i \end{bmatrix}^T \quad (65)$$

It is to be noted that σ_0 may not be explicitly given, but that for a sequence of n orbital positions a $3n \times 3n$ variance-covariance matrix of these positions is given. The method of determing the covariance matrix of the orbital parameters is outlined below.

Consider the parameters $\vec{p} = (\Omega, \omega, I, e, n, \tau)$. By definition, the 6×6 covariance matrix of these parameters, $C(\vec{p})$, is the expected value

$$C(\vec{p}) = E\left\{\begin{bmatrix} d\Omega \\ d\omega \\ di \\ de \\ dn \\ d\tau \end{bmatrix} \begin{bmatrix} d\Omega \\ d\omega \\ di \\ de \\ dn \\ d\tau \end{bmatrix}^T\right\} \tag{66}$$

which may be rewritten as

$$C(\vec{p}) = \left\{\begin{matrix} E\,[d\Omega] \\ E\,[d\omega] \\ E\,[di] \\ E\,[de] \\ E\,[dn] \\ E\,[d\tau] \end{matrix}\right\} \left\{\begin{matrix} E\,[d\Omega] \\ E\,[d\omega] \\ E\,[di] \\ E\,[de] \\ E\,[dn] \\ E\,[d\tau] \end{matrix}\right\}^T \tag{67}$$

These parameters are functionally related to the variables X_i, Y_i, Z_i, and thus each of the individual expectancies may be formulated as

$$E\,\{\vec{p}\} = \frac{\partial p}{\partial X_i}\,E\,\{dX_i\} + \frac{\partial p}{Y_i}\,E\,\{dY_i\} + \frac{\partial p}{Z_i}\,E\,\{dZ_i\} \tag{68}$$

By successive substitution, and factoring it is found that $C\vec{p}$ can be expressed as

$$C(\vec{p})_i = \begin{bmatrix} \partial\Omega/\partial X_i & \partial\Omega/\partial Y_i & \partial\Omega/\partial Z_i \\ \partial\omega/\partial X_i & \partial\omega/\partial Y_i & \partial\omega/\partial Z_i \\ \partial i/\partial X_i & \partial i/\partial Y_i & \partial i/\partial Z_i \\ \partial e/\partial X_i & \partial e/\partial Y_i & \partial e/\partial Z_i \\ \partial n/\partial X_i & \partial n/\partial Y_i & \partial n/\partial Z_i \\ \partial\tau/\partial X_i & \partial\tau/\partial Y_i & \partial\tau/\partial Z_i \end{bmatrix} E\left\{\begin{bmatrix} dX_i \\ dY_i \\ dZ_i \end{bmatrix}\begin{bmatrix} dX_i \\ dY_i \\ dZ_i \end{bmatrix}^T\right\}\begin{bmatrix} \partial\Omega/\partial X_i & \partial\Omega/\partial Y_i & \partial\Omega/\partial Z_i \\ \partial\omega/\partial X_i & \partial\omega/\partial Y_i & \partial\omega/\partial Z_i \\ \partial i/\partial X_i & \partial i/\partial Y_i & \partial i/\partial Z_i \\ \partial e/\partial X_i & \partial e/\partial Y_i & \partial e/\partial Z_i \\ \partial n/\partial X_i & \partial n/\partial Y_i & \partial n/\partial Z_i \\ \partial\tau/\partial X_i & \partial\tau/\partial Y_i & \partial\tau/\partial Z_i \end{bmatrix} \tag{69}$$

Noting that the central factor is the covariance matrix of $X_iY_iZ_i$, denoting the 6×3 Jacobian as $J(\vec{p}; \vec{X})$, then we may express $C(\vec{p})$ in terms of $C(\vec{X}_i)$ according to

$$C(\vec{p})_i = J(\vec{p}; \vec{X}_i)\, C(\vec{X}_i)\, J^T(\vec{p}; \vec{X}_i) \tag{70}$$

It is to be noted that $C(\vec{p})_i$ is a variable, depending on the values of $\vec{X}_i$ and the values of $C(\vec{X}_i)$, which however becomes a constant if $C(\vec{X}_i)$ varies in the appropriate fashion. This implies a dependence between successive values of $\vec{X}_i$, as substantiated by fact. Thus the covariance matrix $C(\vec{p})$ determined from m successive positions may be written as

$$\underset{6 \times 6}{C(\vec{p})} = \underset{6 \times 3m}{J(\vec{p}; \vec{X}_i)}\ \underset{3m \times 3m}{C(\vec{X}_i)}\ \underset{3m \times 6}{J^T(\vec{p}; \vec{X}_i)} \qquad i = 1, \ldots m \tag{71}$$

which is readily soluble provided that the $3m \times 3m$ covariance matrix $C(\vec{X}_i)$ is known. It is noted that Equations (71) and (68) are identical in meaning, since the right hand side of (71) is a summation over m points, which is in accordance with the definition of an expected value.

The two condition equations for each ground point image are:

$$A_i V_i + B_i \Delta + E_i = 0$$

where

$$A_2 = \begin{bmatrix} \underset{(2 \times 6)}{A_{X_i}} & \underset{(2 \times 21)}{0} \\ \underset{(21 \times 6)}{0} & \underset{(21 \times 21)}{I} \end{bmatrix} \tag{72}$$

Now,

$$V_i = [V_{x_i}, V_{y_i}, V_{x_i'}, \ldots\ldots V_T, V_{X_i}, V_{Y_i}, V_{Z_i}]^T \tag{73}$$

$$B_i = \begin{bmatrix} B_2 \\ (2 \times 21) \\ -I \\ (21 \times 21) \end{bmatrix} \tag{74}$$

$$\Delta_i = [\delta_{x_0}, \delta_{y_0}, \delta f, \delta_{x_0'}, \delta_{y_0'}, \delta f', \ldots\ldots \delta Z_i]^T \tag{75}$$

and

$$E_i = [E_{x_i}, E_{y_i}, E_{x_i'}, E_{y_i'}, \ldots\ldots E_{Z_i}]^T \tag{76}$$

The associated weight matrix is

$$\sigma^{-1} = \text{Diagonal}\,[\sigma_{x_i}, \sigma_{x_i'}, \sigma_{x_i''}, \sigma_c, \sigma_c', \sigma_c'', \sigma_0, \sigma_\alpha, \sigma_{X_i}] \tag{77}$$

On forming the normal equations, rewritten as $N\Delta = -C$, one obtains

$$N = B_i^T (A_i \sigma A_i^T)^{-1} B_i \tag{78}$$

and

$$C = B_i^T (A_i \sigma A_i^T)^{-1} E_i \tag{79}$$

from which the solution for Δ is obtained as

$$\Delta = -N^{-1} C \tag{80}$$

The formal solution of (80) indicates that it will be necessary to invert a $(16 + 3_i + 3_j) \times (16 + 3_i + 3_j)$ matrix as a minimum. If the angles ω_j, φ_j, κ_j are expressed in the form $\alpha = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_m t^m$, the order of the matrix is increased by 3_{mj}. This is so formidable a task as to yield it impractical, even for existing computers. Various techniques have been developed for the inversion of large matrices. These are described in Appendix C.

However, if one reconsiders the normal equations, it is found that they consist of two parts—one pertaining to the orbital and camera parameter data, and one relating to the ground point data.

Following the notation and derivation of Brown, the observation Equation (46) may be rewritten in the form

$$AV + \dot{B}\dot{\delta} + \ddot{B}\ddot{\delta} + E = 0 \tag{81}$$

where $\dot{B}$, $\dot{\delta}$ refer to orbital and camera parameters

$\ddot{B}$, $\ddot{\delta}$ refer to the ground point data

Equation (60) may be rewritten in the form

$$W = (A\sigma A^T)^{-1} = \begin{bmatrix} W_X & 0 & 0 \\ 0 & \dot{W} & 0 \\ 0 & 0 & \ddot{W} \end{bmatrix} = \begin{bmatrix} (A_X\sigma_X A^T)^{-1} & 0 & 0 \\ 0 & (\dot{\sigma})^{-1} & 0 \\ 0 & 0 & (\ddot{\sigma})^{-1} \end{bmatrix} \tag{82}$$

where $\dot{\sigma}$ = Diagonal $[\sigma_c, \sigma_c{}', \sigma_c{}'', \sigma, \sigma\alpha_j]$

$\ddot{\sigma}$ = Diagonal $[\sigma X_1, \sigma X_2, \ldots\ldots \sigma X_i]$

Equation (82) is equivalent to

$$W = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \sigma_X{}^{-1} & 0 & 0 \\ 0 & \dot{W} & 0 \\ 0 & 0 & \ddot{W} \end{bmatrix} \begin{bmatrix} A^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Similarly (61) may be rewritten as

$$N = B^T WB = \begin{bmatrix} \dot{B}^T & -I & 0 \\ \ddot{B}^T & 0 & -I \end{bmatrix} \begin{bmatrix} W_X & 0 & 0 \\ 0 & \dot{W} & 0 \\ 0 & 0 & \ddot{W} \end{bmatrix} \begin{bmatrix} \dot{B} & \ddot{B} \\ -I & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} (\dot{B}^T W_X \dot{B} + \dot{W}) & (\dot{B}^T W_X \ddot{B}) \\ (\ddot{B}^T W_X \dot{B}) & (\ddot{B}^T W_X \ddot{B} + \ddot{W}) \end{bmatrix} = \begin{bmatrix} \dot{N} + \dot{W} & \bar{N} \\ \bar{N}^{-T} & \ddot{N} + \ddot{W} \end{bmatrix} \tag{83}$$

Similarly, Equation (62) becomes

$$C = B^T WE = \begin{bmatrix} \dot{B}^T & -I & 0 \\ & & \\ \ddot{B}^T & 0 & -I \end{bmatrix} \begin{bmatrix} W_x & 0 & 0 \\ 0 & \dot{W} & 0 \\ 0 & 0 & \ddot{W} \end{bmatrix} \begin{bmatrix} E_x \\ \dot{E} \\ \ddot{E} \end{bmatrix}$$

or

$$C = \begin{bmatrix} (\dot{B}^T W_x E_x - \dot{W}\dot{E}) \\ \\ (\ddot{B}^T W_x E_x - \ddot{W}\ddot{E}) \end{bmatrix} \hat{=} \begin{bmatrix} \dot{C} - \dot{W}\dot{E} \\ \\ \ddot{C} - \ddot{W}\ddot{E} \end{bmatrix} \tag{84}$$

The normal equations, $N\Delta = -C$, are thus

$$\begin{bmatrix} \dot{N} + \dot{W} & \bar{N} \\ & \\ \bar{N}^T & \ddot{N} + \ddot{W} \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \\ \ddot{\delta} \end{bmatrix} = - \begin{bmatrix} \dot{C} - \dot{W}\dot{E} \\ \\ \ddot{C} - \ddot{W}\ddot{E} \end{bmatrix} \tag{85}$$

which for computational purposes is partitioned:

$$\left[\begin{array}{c|cccc} \dot{N} + \dot{W} & \bar{N}_1 & \bar{N}_2 & \cdots & \bar{N}_i \\ \hline \bar{N}_1^T & \ddot{N}_1 + \ddot{W}_1 & & & \\ \bar{N}_2^T & & \ddot{N}_2 + \ddot{W}_2 & & \\ \bar{N}_i^T & & & & \ddot{N}_i + \ddot{W}_i \end{array}\right] \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_i \end{bmatrix} = \begin{bmatrix} \dot{C} - \dot{W}\dot{E} \\ \ddot{C}_1 - \ddot{W}_1\ddot{E}_1 \\ \ddot{C}_2 - \ddot{W}_2\ddot{E}_2 \\ \ddot{C}_i - \ddot{W}_i\ddot{E}_i \end{bmatrix} \tag{86}$$

Let $N^{-1} = M$, according to

$$\begin{bmatrix} \dot{N} + \dot{W} & \bar{N} \\ & \\ \bar{N}^T & \ddot{N} + \ddot{W} \end{bmatrix}^{-1} = \begin{bmatrix} \dot{M} & \bar{M} \\ & \\ \bar{M}^T & \ddot{M} \end{bmatrix} \tag{87}$$

Since $NM = I$, then, noting that $\bar{M} = -\dot{M}\,\bar{N}\,(\ddot{N} + \ddot{W})^{-1}$, one obtains

$$\dot{M} = [(\dot{N} + \dot{W}) - \bar{N}\,(\ddot{N} + \ddot{W})^{-1}\,N^T]^{-1} \tag{88}$$

and

$$\ddot{M} = [(\ddot{N} + \ddot{W})^{-1} + (\ddot{N} + \ddot{W})^{-1}\,\bar{N}^T\dot{M}\,\bar{N}\,(\ddot{N} + \ddot{W})^{-1}] \tag{89}$$

In order to determine the corrections $\dot{\delta}$, $\ddot{\delta}$, it is not necessary to solve Equation (89). Consider

$$-\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{M} & \bar{M} & \dot{C} - \dot{W}\dot{E} \\ \bar{M}^T & \ddot{M} & \ddot{C} - \ddot{W}\ddot{E} \end{bmatrix} \quad (90)$$

from which

$$-\dot{\delta} = \dot{M}\,(\dot{C} - \dot{W}\dot{E}) + \bar{M}\,(\ddot{C} - \ddot{W}\ddot{E})$$

and

$$-\ddot{\delta} = \bar{M}^T\,(\dot{C} - \dot{W}\dot{E}) + \ddot{M}\,(\ddot{C} - \ddot{W}\ddot{E})$$

Putting

$$Q = (\ddot{N} + \ddot{W})^{-1}\,\bar{N}^T \quad (91)$$

then

$$\bar{M} = -\dot{M}\,Q$$

and

$$-\dot{\delta} = \dot{M}\,(\dot{C} - \dot{W}\dot{E} - Q\,(\ddot{C} - \ddot{W}\ddot{E}) \quad (92)$$

Since

$$-\,[\bar{N}^T\dot{\delta} + (\ddot{N} + \ddot{W})\,\ddot{\delta}] = \ddot{C} - \ddot{W}\ddot{E}$$

then

$$-\,(\ddot{\delta}) = (\ddot{N} + \ddot{W})^{-1}\,(\ddot{C} - \ddot{W}\ddot{E}) - Q\dot{\delta} \quad (93)$$

It is to be noted that $\ddot{N} + \ddot{W}$ consists of $\underline{i}$ diagonally arranged 3×3 matrices, and consequently presents no difficulties. $\dot{M}$ requires the inversion of a $(16 + 3_{mj})$ matrix. For the purpose of this project, this is not considered to be too large; however in the event of large m, j, $\dot{N} + \dot{W}$ may be suitably partitioned.

3.4 ERROR PROPAGATION

In the final iteration

$$V_x = E_x - \dot{B}\,\dot{\delta} - \ddot{B}\,\ddot{\delta}$$

$$\dot{V} = \dot{E} - \dot{\delta}$$

and

$$\ddot{V} = \ddot{E} - \ddot{\delta}$$

with $\dot{\delta}$ and $\ddot{\delta}$ tending to zero, i.e.,

$$V_x \sim E_x$$
$$\dot{V} \sim \dot{E}$$
$$\ddot{V} \sim \ddot{E}$$

Now

$$S = V_x^T W_x\, V + \dot{V}^T \dot{W}\, \dot{V} + \ddot{V}^T \ddot{W}\, \ddot{V} \qquad (94)$$

Equation (94) which when divided by the degrees of freedom gives the unit variance σ_0.

$\dot{M}$ is the covariance matrix of the adjusted camera and orbital parameters, and $\ddot{M}$ is that of the ground points. It is to be noted that $\ddot{M} = (\ddot{N} + \ddot{W})^{-1} + Q\,\dot{M}\,Q^T$, so that the covariance matrix of any point g is

$$\ddot{M}_g = (\ddot{N}_g + \ddot{W}_g)^{-1} + Q_g\,\dot{M}_g\,Q_g^T \qquad (95)$$

3.5 AUXILIARY DATA

Suppose auxiliary data, independent of the camera system, has been collected, which may be expressed as a function of the various parameters. As an example, a radar altimeter will indicate the value of nadiral distance, which may be expressed in terms of the orbital parameters.

Denote the vector of auxiliary data as Λ, which may be written as

$$\Lambda_h = F_h\,(\Omega,\ \omega,\ \ldots\ldots\ \tau,) \qquad (96)$$

then, as before

$$\Lambda_h = \Lambda_h^\circ + V_{\Lambda h} \tag{97}$$

The various parameters $\alpha_p = (\Omega, \omega, \ldots\ldots \tau,)$ may be expressed as

$$\alpha_p = \alpha_p^\circ + \delta\alpha_p \tag{98}$$

so that

$$V_{\Lambda h} - \Lambda_{h_1}\,\delta\alpha_1 \ldots\ldots \Lambda_{hp}\,\delta\alpha_p = E_{\Lambda h} \tag{99}$$

where

$$E_{\Lambda h} = -F_h\,(\alpha_1^\circ, \alpha_2^\circ, \ldots\ldots \alpha_p^\circ) \tag{100}$$

Consequently, may write an additional series of equations:

$$V_\Lambda - \Lambda\dot{\delta} = E_\Lambda \tag{101}$$

which, together with the covariance matrix σ_Λ, may be incorporated into the previous solution, according to

$$\begin{bmatrix} V_x \\ V_\Lambda \\ \\ V \\ V \end{bmatrix} + \begin{bmatrix} B & B \\ -\Lambda & 0 \\ & \\ -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} E_x \\ E_\Lambda \\ \\ \dot{E} \\ \ddot{E} \end{bmatrix} \tag{102}$$

to yield

$$N = \begin{bmatrix} (\dot{B}^T W_x \dot{B} + \dot{W} + \Lambda^T W_\Lambda\,\Lambda) & \dot{B}^T \ddot{W}\,\ddot{B} \\ \\ \ddot{B}^T W_x \dot{B} & (\ddot{B}^T W_x \ddot{B} + \ddot{W}) \end{bmatrix} = \begin{bmatrix} \dot{N} + \dot{W} + \Lambda^T W_\Lambda\,\Lambda^T & \bar{N} \\ \\ \bar{N}^T & \ddot{N} + \ddot{W} \end{bmatrix} \tag{103}$$

and

$$C = \begin{bmatrix} \dot{C} - \dot{W}\dot{E} - \Lambda\, W_\Lambda\, E_\Lambda \\ \\ \ddot{C} - \ddot{W}\ddot{E} \end{bmatrix} \qquad (104)$$

Similarly, auxiliary data pertaining to ground point data, $(X_i\ Y_i\ Z_i)$ might be used to exploit the relationship between auxiliary data and the various parameters.

4. EMPIRICAL SOLUTION

Let it be supposed that it is desired to express the ground coordinates of points as a polynomial, in terms of the panoramic image coordinates, $(x_i - x_0)$, $(y_i - y_0)$. In order to do this with some discrimination, let the projective relationship be expanded. Consider the form

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + K_i M \begin{bmatrix} -f \sin \alpha_i \\ (y_i - y_0) + C \sin \alpha_i \\ -f \cos \alpha_i \end{bmatrix} \tag{105}$$

in which K_i is a variable scale factor, and M is the orientation matrix.

Now, K_i is the ratio of the length of the object space vector to that of the image space vector, which for a tilted photograph is obtained as

$$K_i = H/f \cos t \cos \alpha_i \tag{106}$$

which may be rewritten as

$$K = \frac{H}{f \cos t} [1 + \alpha^2/2! + 5\alpha^4/4! + 61\alpha^6/6! + \dots] \tag{107}$$

in which the suffix i has been supressed.

Denoting the elements of the matrix M as m_{ij}, noting that the suffix i is not the same as was previously used, the component V_i, i = 1,2,3, of the last term of (105) may be rewritten as

$$V_i \sim K \begin{cases} m_{i1} [-f (\alpha^3/3! + \alpha^5/5! - \dots)] \\ m_{i2} [(y_i - y_0) + C (\alpha - \alpha^3/3! + \alpha^5/5! - \dots)] \\ m_{i3} [-f (1 - \alpha^2/2! + \alpha^4/4! - \alpha^6/6! + \dots)] \end{cases} \tag{108}$$

Substituting (107) and (108), and putting k = H/f cos t yields

$$V_i = k \begin{cases} (m_{i2}\, C - m_{i1}\, f)(1 + \alpha^2/2! + 5\alpha^4/4! + 61\alpha^6/6! + \ldots) \\ (\alpha - \alpha^3/3! + \alpha^5/5! - \ldots) \\ +m_{i2}\,(y_i - y_0)(1 + \alpha^2/2! + 5\alpha^4/4! + 61\alpha^6/6! + \ldots) \\ -m_{i3}\, f \end{cases} \qquad (109)$$

This yields the odd ordered polynomial of the form

$$V_i = A + B\,(y_i - y_0) + C\,(x_i - x_0) + D\,(y_i - y_0)(x_i - x_0)^2 + E\,(x_i - x_0)^3 + \ldots \qquad (110)$$

since $\alpha_i = (x_i - x_0)$. Provided that $\alpha_i \leq 90$ degrees, and that sufficient control points are available for a solution, an empirical fitting using Equation (110) will furnish ground coordinate values.

5. EXPERIMENTAL DESIGN FOR AN ERROR MODEL

5.1 INTRODUCTION

The objective of prescribing and adhering to a designed experiment is to determine the effects of various factors on the output data. In practice, the use of real observations dictates that the experimental data are acquired under specified conditions. These conditions are then individually and sequentially varied. These classified data are then subjected to statistical techniques through which the effects and interaction of the varied conditions on the end product may be determined.

Without real data, fictitous data may be simulated from a mathematical model, and treated in the same way. Although this, and Monte Carlo techniques, are frequently the only solution to a specific problem, the present error model lends itself to a more elegant and ecomonic method of analysis. This is discussed in the following sections, in which the following assumptions are made.

1. That there will be four classes of ground control data, the classification being dependent on the precision of these data
2. That sufficient ground control data is recognized on frame photography for the location of the exposure stations
3. That panoramic records may contain zero or more absolute ground control points
4. That panoramic records may be subdivided for mapping purposes into three or four segments, each containing a minimum of nine well distributed photogrammetrically determined control points
5. That the data reduction procedure consists of one of the following combinations:
   a. One frame and one pan photograph
   b. One frame and two stereo pan photographs
   c. One pan photograph and two stereo pan photographs
   d. Two stereo frame and two stereo pan photographs, in which the pan photography always overlaps the frame (stereo) imagery
6. That a sequence of one or more combinations of the preceding variations may be adjusted simultaneously

7. That various constraints in addition to parameters describing the vehicle path may be employed in obtaining a solution

5.2 GENERAL REMARKS

Although the techniques of multi-factorial analysis are well known, and frequently used, the usual approach when using simulated data is to apply perturbations to these data and utilize a Monte Carlo technique. Little, or no use is made of sets of exact data with variations in the weighting functions for these data.

This may be a consequence of a result being presented in the form of a variance-covariance matrix, rather than in the form of the easily understood residual errors. Provided that one has a set of exact data, the error term in the reduced condition equations is always zero. Consequently the vector of parameter corrections and of residual errors must always be zero. Consequently, no matter what weights one assigns to the exact input data, the resulting parameter determination will always furnish the same exact values. They are however, different in the sense that each result is associated with factors applied to the original input material.

Varying the weights of a set of data is far simpler and more economic of computer time than generation random numbers to perturb data prior to a solution. Furthermore, the analysis of the resulting computations is simplified.

The suggested method is to select a set of exact data, from which one may form the weighted normal equations using zero variances for all but one parameter or variable. By assigning a sequence of variances to this parameter, the independent effects of these precisions on the output data is indicated by the resulting variance-covariance matrices. By sequentially varying the combinations of weights, different covariance matrices are obtained, from which a complete analysis of the system can be made.

In order that the various techniques might be better understood, and more readily interpreted, the following section is devoted to some comments on the character of the variance-covariance matrix. For those familiar with covariance techniques, this section may be ignored.

5.3 COVARIANCE METHODS

The covariance of two variables x_i, and x_j is defined as the expected value

$$E\{(x_i - \mu_i)(x_j - \mu_j)\} = E\{(x_i x_j)\} - E\{x_i\} E\{x_j\} = \sigma_i \sigma_j \rho_{ij}$$

where $\mu_{i,j}$ designates the mean value of $x_{i,j}$.

Similarly, the variance of x_i is defined as the expected value

$$E\{(x_i - \mu_i)^2\} = E\{x_i^2\} - (E\{x_i\})^2 = \sigma_i^2$$

Assuming that the observational errors e_i, of a m-variate distribution have a joint distribution with zero means, then $\sigma_i^2 = E\{e_i^2\}$ and $\sigma_i \sigma_j \rho_{ij} = E\{e_i e_j\}$. These are the general terms of a $n \times n$ matrix, M, which is termed the variance-covariance matrix.

$$M = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} = \begin{bmatrix} E\{e^2\} & E\{e_1 e_2\} & \dots & E\{e_1 e_n\} \\ E\{e_2 e_1\} & E\{e_2^2\} & \dots & E\{e_2 e_n\} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ E\{e_n e_1\} & E\{e_n e_2\} & \dots & E\{e_n^2\} \end{bmatrix}$$

This matrix may be normalized by transforming it into the correlation matrix, p, in which the elements p_{ij} are obtained as

$$p_{ij} = \frac{\text{Covariance }(x_i\, x_j)}{[\text{Variance }(x_i)\text{ Variance }(x_j)]^{1/2}} = \frac{m_{ij}}{[m_{ii} \cdot m_{jj}]^{1/2}}$$

Obviously if $i = j$, $p_{ij} = 1$.

For a multi-variate case, these correlation coefficients p_{ij} include the indirect dependence of both x_i and x_j on the n-2 remaining variables. Unless these correlations are extremely small, the linear dependence of x_i and x_j is not obvious. The direct linear dependence of these two variables is given by the partial correlation coefficient

$$p_{ij} \mid (1, \dots n)/ij$$

This is obtained from the normal equation matrix B (the inverse of the covariance matrix) according to

$$p_{ij} \mid (1, \dots n)/ij = - b_{ij}/(b_{ii} \cdot b_{jj})^{1/2}$$

Correlation coefficients of varying orders may also be defined, but are beyond the scope of this section.

It is of interest to consider the conditional variance defined as

$$\sigma_i^2 \mid (1, \dots n)/i = 1/b_{ii}$$

in order that it might be distinguished from the marginal variance obtained from the covariance matrix M, and used in evaluating the multiple correlation coefficient R_i.

Consider the bivariate distribution comprised of the variables x_i, x_j. Suppose that the error ellipse of a point O is represented by Figure 5-1. σ_i, σ_j, are the projection of the ellipse on the axes x_i, x_j and are the marginal standard deviations, whereas the lengths OP, OQ are the conditional standard deviations $\sigma\ (x_1\ x_2)$ and $\sigma\ (x_2\ x_1)$.

The conditional variance is always less than the marginal variance, unless the correlation between x_i and x_j is zero for all correlation coefficient R_i, defined by

$$R_i^2 = 1 - \frac{\sigma_i^2 \mid (1, 2, \ldots n)/i}{\sigma_i^2} = 1 - 1/b_{ii}\,\sigma_i^2$$

which is a measure of the total linear dependence of the variable x_i on the remainder.

5.4 APPLICATIONS AND ANALYSIS

The preliminary requirement for the performance of the system analysis, is that exact data be generated suitable for the combinations listed under assumption 5. These data are readily furnished by the computational sequence listed in Appendix A, Construction of Fictitious Model.

It is to be noted that these calculated data are exact, save for truncation errors. As indicated in the descriptive section, the resulting photo-coordinates are not subject to displacements caused by aberration, atmospheric refraction, lens distortions and the like. This is unimportant, since in practice such displacements would be removed from the measured image coordinates, as far as possible, before being subjected to the main data reduction scheme. It is however, an extremely simple modification to include these displacements should they be required.

From these computed data one obtains a series of exact photo-coordinates, related to the corresponding exact ground coordinates with a transformation involving exact flight and camera parameters. The objective of the subsequent computations is to determine the effects of various inaccuracies in these coordinates and parameters.

To illustrate the method of achieving this end, a consideration of the effects of errors in the measured photo-coordinates on the determination of ground points coordinates, will be made in some detail.

These are four distinct cases listed under assumption 5. Consequently, there will be four sets of parallel computations. To determine the independent effects of errors in the measured photo-coordinates on the end results, the influences of camera and vehicle path parameters are excluded by assinging to each of them, a zero variance.

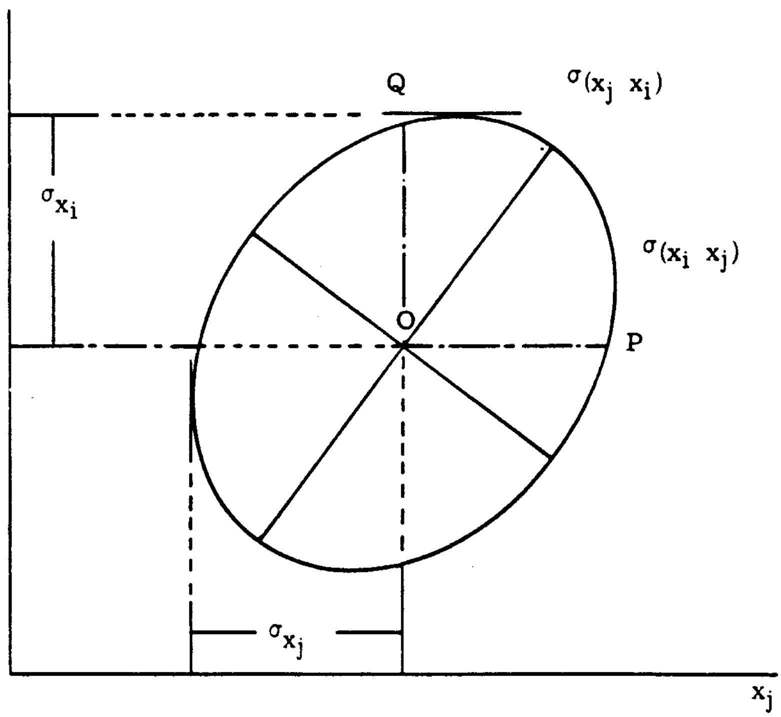


Fig. 5-1 — Error ellipsoid

The measured photo-coordinates are assigned variances (and covariances) proportional to the specified measuring errors under investigation. Since the system parameters are exactly known, together with the transformation relating ground and photo-coordinates, the exact ground control data will be assigned zero weights. In this way, the resulting variance-covariance matrices of the computed ground points reflects only the influence of the errors in the measured image coordinates.

It is suggested that five solutions be obtained for each case corresponding to the selected standard deviations in image measurement of ± 1, ± 2, ± 4, ± 8, and ± 16 microns. These standard deviations are not quite arbitrarily selected, since they correspond to the quoted standard deviations of specific instruments covering the range of envisioned mensuration equipment.

The variance-covariance matrices of the results, is the important output. They indicate the effects of the estimated measuring errors or the ground coordinate determination. Recalling the previous section, the diagonal elements of these matrices represent the marginal variances of the ground coordinates.

In the event that the computational procedure preserves the normal equation matrices, conditional variances, together with the multiple and partial correlation coefficients are determinate. However, it is not usual to preserve the normal equations in a solution by electronic computer, owing to storage requirements. In this event, one can only manipulate with the variance-covariance matrix to furnish correlation coefficients, which indicate the effect of all other parameters and variables on that one which is being investigated.

However, remembering that the partial correlation coefficients reflect the effect of a single parameter on the end result, and that the multiple correlation coefficients indicate the effect of some selected parameters or variables, one realizes that this is exactly what the designed computational procedure accomplishes. Admittedly, if the effects of photographic x-coordinates are to be divorced from those of the associated y-coordinates, one should assign zero variance to that group of coordinates whose influence is to be deleted. This adds two more computational sequences—yielding two partial (x, y) correlation coefficients and one multiple (x y) correlation coefficient. At present, it is not at all apparent what advantages such a refinement would yield, apart from a more complete analysis. This has been mentioned, however, to point out that any specified degree of refinement is easily attained. This provides a method whereby the effects of measuring errors in the panoramic and frame imagery can be separated, leading to valid selections of the appropriate image measuring engines. Furthermore, it indicates that retention of the normal equation matrix is unnecessary for this portion of the system analysis, implying storage for a larger volume of pertinent data.

At the end of the computational sequence, one has amassed a sequence of variance-covariance matrices, and their normalized equivalents, the correlation matrices.

Such numerical data cannot be readily interpreted and evaluated, even by those to whom numerical analysis is second nature.

It would appear that the most striking, and clearly understood, manner of presenting these individual and collective data, is in graphical form. This has the added advantage of being concise and indicative of subtle nuances. The suggested basic presentation illustrated by Figure 5-2, is to present a series of graphs whereby the abscissa represents the standard error of the photo-coordinates, and the ordinate that of the ground coordinates. Correlations, linear, partial, or multiple, are readily presented in the form of regression graphs.

The preceding discussion is readily extended to account for the individual effects of each variable or parameter,* so that the basic portion of the system error analysis may be considered virtually complete, once the sequence of parameter variation has been specified.

This analysis, however, is restricted to a consideration of those factors explicity contained by the transformation formulae. There are certain other factors influencing the results, which one has the tendency to ignore owing to their undefined effects. Of these, perhaps one of the most significant, is the geometric location of control and ground points with respect to camera axes. It would appear that the best manner of performing an analysis of this factor is through a judicious selection of control data and of points to be determined. The results could be suitably presented in the form of a double entry graph—that is with respect to both input variances and angular location of the point.

A similar treatment of the various combinations of ground control data, with respect to both quality and location, will also be performed.

This preliminary error analysis, contained in Table 5-1, is now formulated in some detail, according to the following scheme. It is pointed out that the scheme should be applied to each of the four cases listed under assumption 5, and that it is easily simplified, or modified to include a more complex model.

The numerical data obtained from these calculations and their graphical presentation, provide a sufficiency of data for a thorough system analysis. Furthermore, these data are sufficiently extensive to provide the basis for construction of elegant

---

*Henceforth, parameter is used in the sense of a parameter or variable, unless otherwise indicated. It can be numerically demonstrated that the results of a constrained adjustment are the same whether the unknowns are considered to be parameters or variables. The formal proof of this, has yet to be derived; numerical demonstrations indicate that mathematical induction might be the easiest method of deviation.

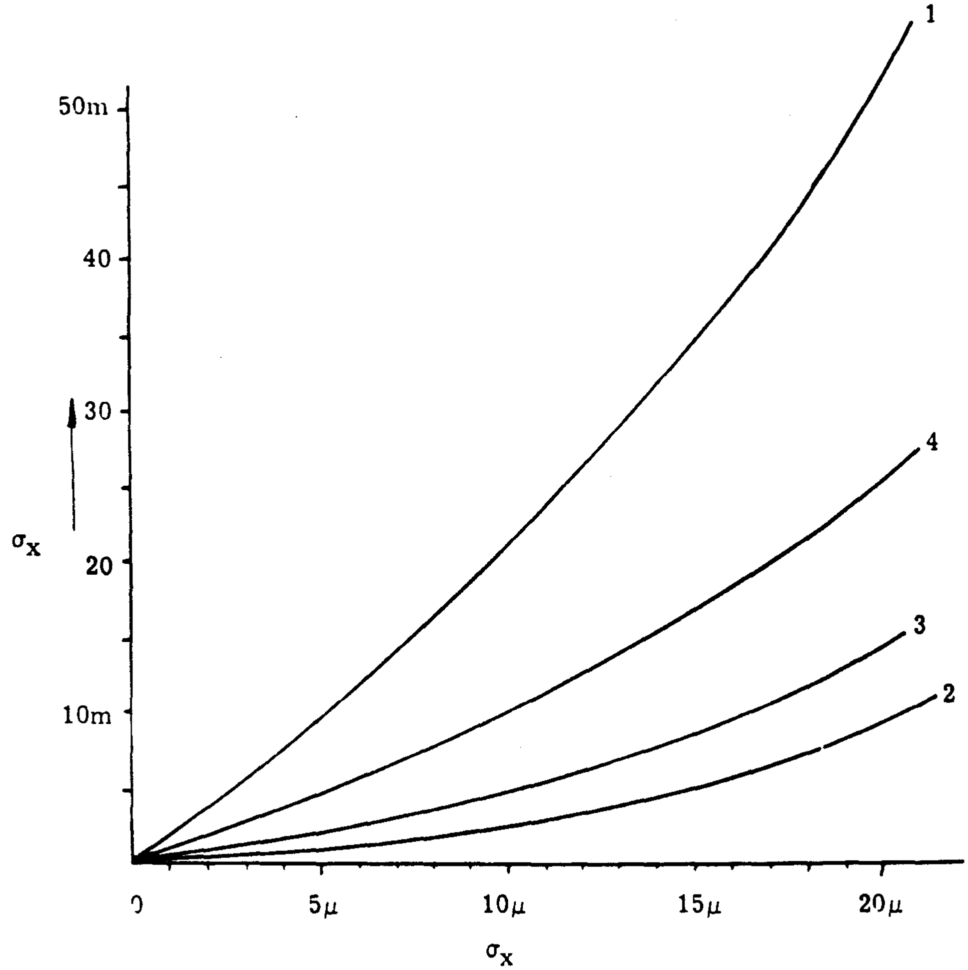


Fig. 5-2 — Standard errors σ_X in position (1), ground X (2), Y (3), and Z (4) coordinates as functions of standard error in photo coordinates (σ_x)

Table 5-1 — Schematic Formulation

Varied Parameters	Remarks
1. Photo-coordinates $(x_1\ y_1)$, (x', y') $(x_2\ y_2)$, (x'', y'')	Vary individually, in combinations according to desired results. Use sequential standard deviations of 1, 2, 4, 8, and 16 microns.
2. Inner orientation elements (x_p, y_p, f), (x_p', y_p', f'), (x_p'', y_p'', f'')	Vary individually, in combinations according to desired results. Use sequential standard deviations of 1, 2, 4, 8, 16, and 32 microns.
3. Angular orientation elements ω_i, ω_i', ω_i''	Vary individually, in combinations according to desired results. Use standard deviations of 1″, 5″, 10″, 30″, 1′, 2′.
4. Angular orientation elements φ_i, φ_i', φ_i''	See 3.
5. Angular orientation elements κ_i, κ_i', κ_i''	See 3.
6. Vehicle path elements Ω, ω, I, e, τ, t	Vary individually, and in combinations. Use a range of variances to cover all expected cases.
7. Pan camera timing $t_{\odot}$, t_i	Vary individually and in combination, using a range of variances to cover expected errors.
8. Ground control quality X_i, Y_i, Z_i	Use 1 all grade I control Use 2 all grade II control Use 3 all grade III control Use 4 all grade IV control
9. Ground control quality (combinations) X_i, Y_i, Z_i	Judiciously select a reasonable sample from all combinations.

Table 5-1 — Schematic Formulation (Cont.)

Varied Parameters	Remarks
10. Ground control location X_i, Y_i, Z_i	Judiciously select a reasonable sample of possible control configurations. Use homogeneous data, in each of four sequences.
11. Ground control location and quality combinations	Use samples 9 and 10 to construct a resultant sample for analysis.
12. Photogrammetric parameters and ground control	Assign zero variances to vehicle path elements. Use reasonable expected values for errors in other parameters, and resonable control distribution.
13. All system parameters	Assign reasonable expected variances to all system parameters.

prediction and control nomograms. Although the basic system error analysis has now been adequately defined, it is thought that the ever recurrent theme of relative v absolute accuracy should be discussed. This is the topic of the succeeding section.

5.5 RELATIVE AND ABSOLUTE ACCURACY CONSIDERATIONS

It is frequently noted that the use of rigorous error propagation techniques using closely estimated variances leads to surprisingly large system errors. As an example, if the well known error equations derived by Brandenbeger are applied to a convergent strip camera system, which has an extremely high resolution, it appears as if precise mapping from these records is not possible. This is a consequence of absolute position and orientation errors of the complete system being quadratically combined, so that the variance that must be assigned to a small just detectable length may be ten or even one hundred times the length itself. Although it is possible to detect and measure the images of small objects with great precision, it is difficult to maintain any degree of accuracy. This difference, between relative and absolute accuracies, epitomizes the different usages of imagery by interpreters and cartographers. In order that one might determine this relative accuracy it will be necessary to consider every parameter to be exact, (in a two photo-mode), save five elements which are necessary to determine the relative orientation between two pictures. Consequently, in a n photosystem it will be necessary to maintain exact values for all but $3n - 1$ parameters. For convenience, it is recommended that the angular orientation elements, ω_i, φ_i, κ_i be those that are varied—being virtually independent of any other parameters.

Appendix A

CONSTRUCTION OF FICTITIOUS MODEL

1.0 INTRODUCTION

This aspect of the problem is to compute exact data by means of which the methods of data reduction and analysis might be tested, and computer programs optimized. The construction of these data falls naturally into four sections.

2.0 DETERMINATION OF SATELLITE POSITION AT ANY TIME t_i

For convenience times will be assumed to be given in Greenwich Apparent Sidereal Time. Although the time measurement may be in some other system, conversions are simple, but necessary.

Given Data

Orbital Parameters: Ω, ω, I, e, n, τ

Geophysical Constants: $\mu = GM = 3.98603 \times 10^{20}$ cm^3/sec^2 (Kaula 1961)

Observed Time: t_i (in GAST)

Computation

1. Determine semi-major axis, a, of orbit

$$a = (\mu \, n^{-2})^{1/3} \qquad (111)$$

2. Determine mean anomaly, M_i, at time t_i:

   a. Compute

$$\chi_i = - n \tau \qquad (112)$$

   b. Compute

$$M_i = nt_i + \chi_i \qquad (113)$$

3. Determine eccentric anomaly, E_i, at time t_i

$$E_i = M_i + e \sin E_i \tag{114}$$

This is performed iteratively according to

$$M_i = E_i^\circ \tag{115}$$

which is substituted into

$$E_i^{\circ\circ} = M_i^\circ + e \sin E_i^\circ$$

4. Determine the Geocentric Terrestrial Coordinates of the Satellite position

$\vec{r}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}_T$, according to

$$\vec{r}_i = a \begin{bmatrix} \cos t_i & \sin t_i & 0 \\ -\sin t_i & \cos t_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Omega & -\cos I \sin \Omega \\ \sin \Omega & \cos I \cos \Omega \\ 0 & \sin I \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix} \begin{bmatrix} -e + \cos(E_i) \\ (1-e^2)^{1/2} \sin E_i \end{bmatrix} \tag{116}$$

5. Determine the heading, geocentric latitude, and longitude of the nadir

$$\cos \lambda_0 = X_i \left(X_i^2 + Y_i^2\right)^{-1/2} \tag{117}$$

$$\sin \lambda_0 = Y_i \left(X_i^2 + Y_i^2\right)^{-1/2} \tag{118}$$

$$\cos \Phi_0 = \left(X_i^2 + Y_i^2\right)^{-1/2} \left(X_i^2 + Y_i^2 + Z_i^2\right)^{-1/2} \tag{119}$$

$$\sin \Phi_0 = Z_i \left(X_i^2 + Y_i^2 + Z_i^2\right)^{-1/2} \tag{120}$$

$$\cos A_0 = \sin I \cos (\lambda_0 - \Omega) \tag{121}$$

$$\sin A_0 = \cos I / \cos \Phi_0 \tag{122}$$

3.0 DETERMINATION OF CAMERA DATA

<u>Given Data:</u> Camera half angle $\alpha/2$

Ellipsoidal semi-major axis $\underset{\sim}{a}$

Ellipsoidal eccentricity $\underset{\sim}{e}$

Flying height H

Camera roll, pitch, ωj, φj

<u>Preliminary Computation</u> - frame cameras

1. Determine flying height H

$$H = | r_i | - R \qquad (123)$$

(R = radius of earth)

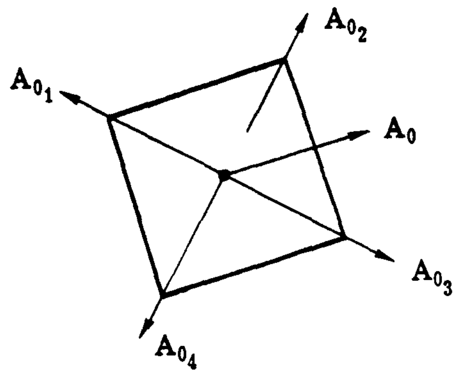
2. Calculate the azimuth to each of the four corners of the imaged terrain from the nadir point

$$A_{0_1} = A_0 + 5\Pi/4 = \alpha_1$$

$$A_{0_2} = A_0 + 7\,\Pi/4 = \alpha_2$$

$$A_{0_3} = A_0 + \Pi/4 = \alpha_3 \qquad (124)$$

$$A_{0_4} = A_0 + 3\,\Pi/4 = \alpha_4$$



3. Compute geodetic latitude of nadir, Ψ_0, from geocentric Φ_0

$$\sin\Psi_0 = \tan\Phi_0[\tan^2\Phi_0 + (1 - \underset{\sim}{e}^2)^2]^{-1/2} \qquad (125)$$

4. Compute radii of curvature at nadir

$$N_0 = \underset{\approx}{a}(1 - \underset{\sim}{e}^2 \sin^2\Psi_0)^{-1/2} \qquad (126)$$

$$R_0 = \underset{\sim}{a}(1 - \underset{\sim}{e}^2)(1 - \underset{\sim}{e}^2 \sin^2\Psi_0)^{-3/2} \qquad (127)$$

5. Compute mean radii of curvature

$$R\alpha_1 = R\alpha_3 = \frac{N_0\, R_0}{R_0 \sin^2 \alpha, + N_0 \cos^2 \alpha_2} \qquad (128)$$

and

$$R\alpha_2 = R\alpha_4 = \frac{N_0\, R_0}{R_0 \sin^2 \alpha_2 + N_0 \cos^2 \alpha_2} \qquad (129)$$

6. Compute the angle θ_α subtended by the arc from the nadir to four corners according to

$$\theta_{\alpha_i} = \sin^{-1}\left[\frac{(R_0 + H) \sin \alpha/2}{R_i}\right] - \alpha/2 \qquad (130)$$

where $\alpha/2$ is the diagonal half angle of the photography.

7. Compute the arc length S_{α_i} according to

$$S_{\alpha_i} = R_{\alpha_i}\, \theta_{\alpha_i}, \ (\theta_{\alpha_i} \text{ in radians}) \qquad (131)$$

8. Compute geodetic coordinates of the four corners according to Rainsford's modification of Clarke's formula:

a) $$\Psi_0' = \Psi_0 + \frac{S_{\alpha_i} \cos \alpha_i}{R_0 \sin 1''} \qquad (132)$$

b) $$\Psi = \frac{\Psi_0 + \Psi_0'}{2} \qquad (133)$$

c) compute R_Ψ:

$$R_\Psi = \underset{\sim}{a}\,(1 - \underset{\sim}{e}^2)(1 - \underset{\sim}{e}^2 \sin^2 \Psi)^{-3/2} \qquad (134)$$

d) compute

$$p_i = \frac{S^2_{\alpha_i} \sin \alpha_i \cos \alpha_i}{2\, R_0\, N_0 \sin 1''} \qquad (135)$$

e) compute

$$q_i = p_i \tan \alpha_i \tan \Psi_0' \tag{136}$$

f) compute

$$\Psi_i = \Psi_0 + \left[\frac{S_{\alpha_i} (\cos \alpha_i - 2p_i/3)}{R_\Psi \sin 1''} - q_i \right] \tag{137}$$

g) compute

$$N_{\Psi_i} = \underset{\sim}{a} (1 - e^2 \sin^2 \Psi_i)^{-1/2} \tag{138}$$

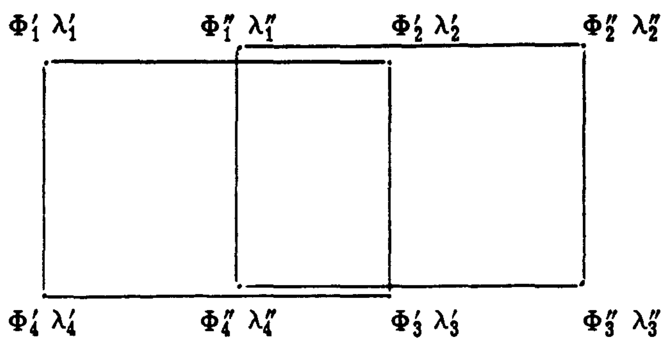
h) compute

$$\lambda_i = \lambda_0 + \left[\frac{S_{\alpha_i} \sin (\alpha_i - 1p_i/3)}{\cos (\Psi_i + q_i/3) N_{\Psi_i} \sin 1''} \right] \tag{139}$$

These values Ψ_i, λ_i define the limits within which the control data for each frame camera are to be determined.

This sequence of computation is performed for both frame cameras.

For each frame camera, designate the corner points of coverage as Φ_1', λ_1', Φ_2', λ_2', Φ_3', λ_3', Φ_4', λ_4'; Φ_1'', λ_1'', Φ_2'', λ_2'', Φ_3'', λ_3'', Φ_4'', λ_4''; as in the following diagram:



Put $\Phi_1', \lambda_1' = \Phi_1, \lambda_1$

$\Phi_2'', \lambda_2'' = \Phi_2, \lambda_2$

HANDLE VIA
TALENT-KEYHOLE
NTROL SYSTEM

$$\Phi_3'', \lambda_3'' = \Phi_3, \lambda_3$$

$$\Phi_4'', \lambda_4'' = \Phi_4, \lambda_4$$

Then the area covered by the two photographs is bounded by the points Φ_1, λ_1; Φ_2, λ_2; Φ_3, λ_3; Φ_4, λ_4.

Let this area be evenly covered with geodetic control points, comprising I evenly spaced rows and J evenly spaced columns.

Then any control point Φ_{ij}, λ_{ij} has coordinates of

$$\Phi_{ij} = \Phi_1 - [(2_j - 1)(\Phi_1 - \Phi_2)/2J] - [(2_i - 1)(\Phi_1 - \Phi_4)/2I] + [(2_i - 1)(2_j - 1)(\Phi_1 - \Phi_2 + \Phi_3 - \Phi_4)/4IJ] \quad (140)$$

and

$$\lambda_{ij} = \lambda_1 - [(2_j - 1)(\lambda_1 - \lambda_2)/2J] - [(2_i - 1)(\lambda_1 - \lambda_4)/2I] + [(2_i - 1)(2_j - 1)(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)/4IJ] \quad (141)$$

4.0 SPACING OF CONTROL

Although it is not essential for a super abundance of fictitious data to be generated, it is desirable from the point of having superfluous data for statistical purposes.

It has been stated that each of the panoramic photographs (models) may be divided into three abutting segments, each of which will contain a minimum of 5 control points. It is considered desirable that there be three columns of control data, one central column and two lateral, according to the following diagram. Furthermore, there will be seven panoramic models per frame model, which are assumed to be butt joined, dictating that 21 columns of control per overlap be established.

Let it be assumed that the panoramic photography has been rectified to fit the frame photographs. If the format size of the frame is $a \times a$ mm^2, with p-percent overlap, then the width of the pan model is

$$w = \frac{1}{7} \cdot \frac{p}{100} \cdot a \text{ mm}$$

[redacted]

[redacted]

The spacing between each column of control data is

$$\frac{w}{3} = \frac{p}{2100} \cdot a \text{ mm}$$

and the total number of control points per row is

$$J = \left\{ \frac{100}{p} + \frac{p - 100}{100} \right\} \times 21$$

Assuming $p = 60$ percent, $J = 44$

The interval between each control point is approximately 6 to 7 millimeters, for a 9 by 8 inch format.

Arbitrarily, select the rows of control data 10 millimeters apart, to yield $I = 23$ for a 9 by 9 inch format, i.e.,

$$I = \frac{a}{10} \text{ mm}$$

This leads to 1012 points for the covered area.

5.0 COMPUTATION OF PHOTO COORDINATES

5.1 Frame Cameras

1. Compute Φ_{ij}, λ_{ij} for each control point within the limits according to Equations (141) and (142)

2. Generate $I \times J$ random integers between 0 and 100 meters. Associate these with the Φ_{ij}, λ_{ij} in sequence, to represent the point elevation, H_{ij}, above the reference ellipsoid

3. For the frame exposure times t_1, t_2, in Greenwich apparent sidereal time, compute the geocentric sidereal coordinates, according to

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{ij} = \begin{bmatrix} \cos t_i & -\sin t_i & 0 \\ \sin t_i & \cos t_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (N + H) & \cos \varphi \cos \lambda \\ (N + H) & \cos \varphi \sin \lambda \\ (1 + e^2) N + H & \sin \varphi \end{bmatrix}_{ij} \quad (142)$$

4. Compute the geocentric sidereal positions of each camera station according to

[redacted]

[redacted]

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}_{t_i} = a \begin{bmatrix} \cos \Omega & - \cos I \sin \Omega \\ \sin \Omega & \cos I \cos \Omega \\ 0 & \sin I \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1 - e^2)^{1/2} \end{bmatrix} \begin{bmatrix} \cos E - e \\ \sin E \end{bmatrix} \quad (143)$$

as outlined in paragraph 2, this section

5. For each exposure station, compute the converted photo-coordinates of each point, according to

$$x_i = x_0 - f \left[\frac{m_{11} (X_i - X_0) + m_{12} (Y_i - Y_0) + m_{13} (Z_i - Z_0)}{m_{31} (X_i - X_0) + m_{32} (Y_i - Y_0) + m_{33} (Z_i - Z_0)} \right] \quad (144)$$

and

$$y_i = y_0 - f \left[\frac{m_{21} (X_i - X_0) + m_{22} (Y_i - Y_0) + m_{23} (Z_i - Z_0)}{m_{31} (X_i - X_0) + m_{32} (Y_i - Y_0) + m_{33} (Z_i - Z_0)} \right] \quad (145)$$

in which the values m_{kl} are given in Appendix B, detailed formulation, Equations (159) through (167)

6. Reject all points which fall outside the desired format limits

7. It is noted that these are corrected photo-coordinates. If desired, apply displacements due to aberration, lens distortion, atmosphere refraction, etc.

5.2 Pan Cameras

Consider the equivalent frame camera coordinates given by (145), (144) in the previous section, rewritten as

$$x_i = -f \frac{U_i}{W_i} \quad (146)$$

and

$$y_i = -f \frac{V_i}{W_i} \quad (147)$$

noting that

$$x_0 = y_0 = 0$$

Let the equivalent frame be exposed at the mid-point of the scan, at t_{0i}. Any control point Φ_{ij}, λ_{ij} is scanned at some other time t_i. By selecting an approximate value of $t_i = t_i^\circ$, approximate values of

$$x_i = x_i^\circ$$
$$y_i = y_i^\circ$$
$$U_i = U_i^\circ$$
$$V_i = V_i^\circ$$
$$W_i = W_i^\circ$$

may be determined. Consequently one may write

$$t_i = t_i^\circ - \frac{x_i^\circ}{\partial(x_i^\circ)/\partial t}$$

$$t_i = t_i^\circ - \frac{y_i^\circ}{\partial(y_i^\circ)/\partial t}$$

$$t_i = t_i^\circ - \frac{U_i^\circ}{\partial(U_i^\circ)/\partial t} \quad \ldots \text{etc.}$$

Since y_i is the most rapidly changing function with respect to time, use

$$t_i = t_i^\circ - \frac{y_i^\circ}{\partial(y_i^\circ)/\partial t} \tag{148}$$

as the control equation. This is rewritten as

$$t_i = t_i^\circ - (V_i^\circ/W_i^\circ)(\partial(V_i^\circ/W_i^\circ)/\partial t)^{-1} \tag{149}$$

in which

$$\partial (V_i^\circ/W_i^\circ) = 1/W_i^\circ \begin{cases} \dfrac{\partial m_{21}}{\partial t}(X_i - X_0) + \dfrac{\partial m_{22}}{\partial t}(Y_i - Y_0) + \dfrac{\partial m_{23}}{\partial t}(Z_i - Z_0) \\ + m_{21}\dfrac{\partial X_i}{\partial t} + m_{22}\dfrac{\partial Y_i}{\partial t} + m_{23}\dfrac{\partial Z_i}{\partial t} \\ - m_{21}\dfrac{\partial X_0}{\partial t} - m_{22}\dfrac{\partial Y_0}{\partial t} - m_{23}\dfrac{\partial Z_0}{\partial t} \\ + y_i^\circ \left[\begin{array}{l} \dfrac{\partial m_{31}}{\partial t}(X_i - X_0) + \dfrac{\partial m_{32}}{\partial t}(Y_i - Y_0) + \dfrac{\partial m_{33}}{\partial t}(Z_i - Z_0) \\ + m_{31}\left|\dfrac{\partial X_i}{\partial t} - \dfrac{\partial X_0}{\partial t}\right| + m_{32}\left|\dfrac{\partial Y_i}{\partial t} - \dfrac{\partial Y_0}{\partial t}\right| + m_{33}\left|\dfrac{\partial Z_i}{\partial t} - \dfrac{\partial Z_0}{\partial t}\right| \end{array} \right] \end{cases} \quad (150)$$

In this equation

$$\frac{\partial X_i^\circ}{\partial t} = -a\left[(\cos\omega\cos\Omega - \sin\Omega\sin\omega\cos I)\sin(E - e) + (1 - e^2)^{1/2}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos I)\cos E\right]\frac{n}{1 - e\cos E} \quad (151)$$

$$\frac{\partial Y_i^\circ}{\partial t} = -a\left[(\cos\omega\cos\Omega + \cos\Omega\sin\omega\cos I)\sin(E - e) + (1 - e^2)^{1/2}(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos I)\cos E\right]\frac{n}{1 - e\cos E} \quad (152)$$

$$\frac{\partial Z_i^\circ}{\partial t} = -a\left[(\sin I\sin\omega\sin(E - e) - (1 - e^2)^{1/2}\sin I\cos\omega\cos E\right]\frac{n}{1 - e\cos E} \quad (153)$$

$$\frac{\partial X_i}{\partial t} = -X_i\sin t_i - Y_i\cos t_i \quad (154)$$

$$\frac{\partial Y_i}{\partial t} = -X_i\cos t_i - Y_i\sin t_i \quad (155)$$

$$\frac{\partial Z_i}{\partial t} = 0 \quad (156)$$

The various values of $\partial m_{kl}/\partial t$ are obtained as

$$\frac{\partial m_{kl}}{\partial t} = \frac{\partial m_{kl}}{\partial \omega} \cdot \frac{\partial \omega_j}{\partial t} + \frac{\partial m_{kl}}{\partial \varphi_j} \cdot \frac{\partial \varphi_j}{\partial t} + \frac{\partial m_{kl}}{\partial \kappa_j} \cdot \frac{\partial \kappa_j}{\partial t} \tag{157}$$

Since the orientation angles are expressed in the form of

$$\begin{aligned} \omega_j &= \omega_0 + \omega(t_i - t_0) \\ \varphi_j &= \varphi_0 + \varphi(t_i - t_0) \\ \kappa_j &= \kappa_0 + \kappa\,(t_i - t_0) \end{aligned} \tag{158}$$

the terms $\partial\omega_j/\partial t$, $\partial\varphi_j/\partial t$, $\partial\kappa_j/\partial t$ are evaluated as in ω, φ, κ, respectively.

The expressions $\partial m_{kl}/\partial \alpha_j$, $\alpha_j = \omega_j$, φ_j, or κ_j are given by Equations (207) through (233), Appendix B, which require the appropriate values of ω_j, φ_j, κ_j as given by Equation (158), above.

The specific computational procedure is:

1. Assume t_i° ($\neq t_{0i}$)

2. Determine X_i, Y_i, Z_i according to (142), for t_i°

3. Compute X_0, Y_0, Z_0 at time t_i° according to Equation (143)

4. Evaluate m_{kl}, according to Equations (98) through (108), Section 4, using $\alpha_j = \alpha_0 + \alpha\,(t_i^\circ - t_{0i})$

5. Compute V_i°, W_i° and Y_i°, according to Equation (147)

6. Compute $\partial X_i^\circ/\partial t$, ... $\partial Z/\partial t$, according to Equations (151) through (156)

7. Compute $\partial m_{kl}/\partial \alpha_j$, according to Equations (207) through (233), Appendix B

8. Compute the six equations $\partial m_{kl}/\partial t$ according to Equation (156)

9. Compute $\partial(V_i^\circ/W_i^\circ)/\partial t$ according to Equation (150)

10. Evaluate Equation (149):

$$t_i^{\circ\circ} = t_i^\circ - [\,\partial(V_i^\circ/W_i^\circ)/\partial t\,]^{-1}\,(V_i^\circ/W_i^\circ)$$

11. Put $t_i^{\circ\circ} = t_i^{\circ}$

12. Repeat steps 2 . . . 11 until the value of

$$(V_i^{\circ}/W_i^{\circ})\ (\partial\ (V_i^{\circ}/W_i^{\circ})/\partial t)^{-1}$$

is negligibly small.

This is done for each of the points on each of the equivalent frame pictures. These points are then transformed into the appropriate coordinates x_i', y_i', according to Eqs. (19) and (20) in Section 2.

Appendix B

DETAILED FORMULATION

The terms of the orientation matrix $M = M_jO_j$, are suppressing the subscript j on all except the rotation ω_j.

$$m_{11} = -\cos\lambda_0 \{[\cos\kappa \cos\varphi \cos A_0 + (\sin\kappa \cos\omega_j + \cos\kappa \sin\varphi \sin\omega_j) \sin A_0] \sin\Phi_0 - (\sin\kappa \sin\omega_j - \cos\kappa \sin\varphi \cos\omega_j) \cos\Phi_0\} + [\cos\kappa \cos\varphi \sin A_0 + (\sin\kappa \cos\omega_j + \cos\kappa \sin\varphi \sin\omega_j) \cos A_0] \sin\lambda_0 \tag{159}$$

$$m_{21} = -\cos\lambda_0 \{[-\sin\kappa \cos\varphi \cos A_0 + (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \sin A_0] \sin\Phi_0 - (\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j) \cos\Phi_0\} + \sin\lambda_0 [\sin\kappa \cos\varphi \sin A_0 + (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \cos A_0] \tag{160}$$

$$m_{31} = -\cos\lambda_0 \{[\sin\varphi \cos A_0 - \cos\varphi \sin\omega_j \sin A_0] \sin\Phi_0 - \cos\varphi \cos\omega_j \cos\Phi_0\} + \sin\lambda_0 [-\sin\varphi \sin A_0 - \cos\varphi \sin\omega_j \cos A_0] \tag{161}$$

$$m_{12} = -\sin\lambda_0 \{[\cos\kappa \cos\varphi \cos A_0 + (\sin\kappa \cos\omega_j + \cos\kappa \sin\varphi \sin\omega_j) \sin A_0] \sin\Phi_0 - (\sin\kappa \sin\omega_j - \cos\kappa \sin\varphi \cos\omega_j) \cos\Phi_0\} - \cos\lambda_0 [-\cos\kappa \cos\varphi \sin A_0 + (\sin\kappa \cos\omega_j + \cos\kappa \sin\varphi \sin\omega_j) \cos A_0] \tag{162}$$

$$m_{22} = -\sin\lambda_0 \{[-\sin\kappa \cos\varphi \cos A_0 + (\cos\kappa \cos\omega_j - \sin\kappa \sin\omega_j \sin\varphi) \sin A_0] \sin\Phi_0 - (\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j) \cos\Phi_0\} - \cos\lambda_0 [\sin\kappa \cos\varphi \sin A_0 + (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \cos A_0] \tag{163}$$

$$m_{32} = -\sin\lambda_0\,[(\sin\varphi\cos A_0 - \cos\varphi\sin\omega_j\sin A_0)\sin\Phi_0 - \cos\varphi\cos\omega_j\cos\Phi_0] \\ -\cos\lambda_0\,[-\sin\varphi\sin A_0 - \cos\varphi\sin\omega_j\cos A_0] \quad (164)$$

$$m_{13} = [\cos\kappa\cos\varphi\cos A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\sin A_0]\cos\Phi_0 \\ + (\sin\kappa\sin\omega_j - \cos\kappa\sin\varphi\cos\omega_j)\sin\Phi_0 \quad (165)$$

$$m_{23} = [-\sin\kappa\cos\varphi\cos A_0 + (\cos\kappa\cos\omega_j - \sin\kappa\sin\varphi\sin\omega_j)\sin A_0]\cos\Phi_0 \\ + (\cos\kappa\sin\omega_j + \sin\kappa\sin\varphi\cos\omega_j)\sin\Phi_0 \quad (166)$$

$$m_{33} = (\sin\varphi\cos A_0 - \cos\varphi\sin\omega_j\sin A_0)\cos\Phi_0 + \cos\varphi\cos\omega_j\sin\Phi_0 \quad (167)$$

In which

$$\cos\lambda_0 = X_0/(X_0^2 + Y_0^2)^{1/2} \qquad \cos\Phi_0 = (X_0^2 + Y_0^2)^{1/2}/(X_0^2 + Y_0^2 + Z_0^2)^{1/2}$$
$$\sin\lambda_0 = Y_0/(X_0^2 + Y_0^2)^{1/2} \qquad \sin\Phi_0 = Z_0/(X_0^2 + Y_0^2 + Z_0^2)^{1/2} \quad (168)$$
$$\cos A_0 = \sin I\cos(\lambda_0 - \Omega) \qquad \sin A_0 = \cos I/\cos\Phi_0$$

$$\partial F_1/\partial x_i = 1 \qquad \partial F_2/\partial x_i = 0$$
$$\partial F_1/\partial y_i = 0 \qquad \partial F_2/\partial y_i = 1$$
$$\partial F_1/\partial x_0 = -1 \qquad \partial F_2/\partial x_0 = 0 \quad (169)$$
$$\partial F_1/\partial y_0 = 0 \qquad \partial F_2/\partial y_0 = -1$$
$$\partial F_1/\partial f = U_i/W_i \qquad \partial F_2/\partial f = V_i/W_i$$

where

$$U_i = m_{11}(X_i - X_0) + m_{12}(Y_i - Y_0) + m_{13}(Z_i - Z_0) \quad (170)$$

$$V_i = m_{21}(X_i - X_0) + m_{22}(Y_i - Y_0) + m_{23}(Z_i - Z_0) \quad (171)$$

$$W_i = m_{31}(X_i - X_0) + m_{32}(Y_i - Y_0) + m_{33}(Z_i - Z_0) \quad (172)$$

For pan camera it is noted that it is more appropriate to use the expressions

$$F_1 = \tan\alpha - \frac{U_i}{W_i}$$

and

$$F_2 = (y - y_0) + f' \frac{V_i}{W_i}$$

whence one obtains

$$\begin{aligned}
&\partial F_1/\partial x_i = - \sec^2 \alpha_i/f' && \partial F_2/\partial x_i = - C\,[1 + (y' - y_0') \tan^2 \alpha_i]/f' \\
&\partial F_1/\partial x_0 = + \sec^2 \alpha_i/f' && \partial F_2/\partial x_0 = + C\,[1 + (y' - y_0') \tan^2 \alpha_i]/f' \\
&\partial F_1/\partial y_i = 0 && \partial F_2/\partial y_i = + \sec \alpha_i \qquad (173) \\
&\partial F_1/\partial y_0 = 0 && \partial F_2/\partial y_0 = - \sec \alpha_i \\
&\partial F_1/\partial f' = - \alpha_i \sec^2 \alpha_i/f' && \partial F_2/\partial f' = (V/W) - \alpha_i\,[C - (y' - y_0') \sin \alpha_i]/f'\cos^2 \alpha_i
\end{aligned}$$

Note that although

$$C = \frac{V}{h} \cdot \frac{f}{\alpha}$$

C is not a variable in the expression F_2, and must be considered invariant. Furthermore, it is important to note that for a pan camera, the factor (f') assuming the subsequent partial derivatives should be made equal to unity.

Working in sidereal system, i.e., put

$$X_i = X_i \cos t_i - Y_i \sin t_i\ ,\ Y_{is} = X_i \sin t_i + Y_i \cos t_i\ ,\ Z_{is} = Z_i$$

$$\frac{\partial F_1}{\partial X_{is}} = \frac{+f}{W}\left[m_{11} \cos t_i + m_{12} \sin t_i - \frac{U}{W}(m_{31} \cos t_i + m_{32} \sin t_i)\right] \qquad (174)$$

$$\frac{\partial F_1}{\partial Y_{is}} = \frac{+f}{W}\left[-m_{11} \sin t_i + m_{12} \cos t_i - \frac{U}{W}(-m_{31} \sin t_i + m_{32} \cos t_i)\right] \qquad (175)$$

$$\frac{\partial F_2}{\partial Z_{is}} = \frac{+f}{W}\left[m_{13} - \frac{U}{W} m_{33}\right] \qquad (176)$$

$$\frac{\partial F_2}{\partial X_{is}} = \frac{+f}{W}\left[+m_{21} \sin t_i + m_{22} \sin t_i - \frac{V}{W}(+m_{31} \cos t_i + m_{32} \sin t_i)\right] \qquad (177)$$

$$\frac{\partial F_2}{\partial Y_{is}} = \frac{+f}{W}\left[-m_{21}\sin t_i + m_{22}\cos t_i - \frac{V}{W}(-m_{31}\sin t_i + m_{32}\cos t_i)\right] \quad (178)$$

$$\frac{\partial F_2}{\partial Z_{is}} = \frac{+f}{W}\left(m_{23} - \frac{V}{W} m_{33}\right) \quad (179)$$

Define

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{bmatrix} \begin{bmatrix} \cos(E) - e \\ \sin E \end{bmatrix} = R_0 \quad (180)$$

$$\begin{aligned} A_x &= a(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos I) \\ A_y &= a(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos I) \\ A_z &= a(\sin I\sin\omega) \\ B_x &= -a(1-e^2)^{1/2}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos I) \\ B_y &= -a(1-e^2)^{1/2}(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos I) \\ B_z &= a(1-e^2)^{1/2}(\sin I\cos\omega) \end{aligned} \quad (181)$$

$$\frac{\partial X^\circ}{\partial\Omega} = -a\{(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos I)[\cos(E) - e] - (1-e^2)^{1/2}(\sin\Omega\sin\omega - \cos\Omega\cos\omega - \cos\Omega\cos\omega\cos I)\sin E\} \quad (182)$$

$$\frac{\partial Y^\circ}{\partial\Omega} = -a\{(-\cos\Omega\cos\omega + \sin\Omega\sin\omega\cos I)[\cos(E) - e] + (1-e^2)^{1/2}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos I)\sin E\} \quad (183)$$

$$\frac{\partial Z^\circ}{\partial\Omega} = 0 \quad (184)$$

$$\frac{\partial X^\circ}{\partial\omega} = -a\{(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos I)[\cos(E) - e] + (1-e^2)^{1/2}(\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos I)\sin E\} \quad (185)$$

$$\frac{\partial Y^\circ}{\partial \omega} = -a \{(\sin \Omega \sin \omega - \cos \Omega \cos \omega \cos I) [\cos (E) - e]$$

$$+ (1-e^2)^{1/2} (\sin \Omega \cos \omega \cos \Omega \sin \omega \cos I) \sin E\} \qquad (186)$$

$$\frac{\partial Z^\circ}{\partial \omega} = +a \{\sin I \cos \omega [\cos (E) - e] - \sin I \sin \omega \sin E (1-e^2)^{1/2}\} \qquad (187)$$

$$\frac{\partial X^\circ}{\partial I} = a\{\sin \Omega \sin \omega \sin I [\cos (E) - e] + (1-e^2)^{1/2} \sin \Omega \cos \omega \sin I \sin E\} \qquad (188)$$

$$\frac{\partial Y^\circ}{\partial I} = a \{-\cos \Omega \sin \omega \sin I [\cos (E) - e] - (1-e^2)^{1/2} \cos \Omega \cos \omega \sin I \sin E\} \qquad (189)$$

$$\frac{\partial Z^\circ}{\partial I} = a \{\cos I \sin \omega [\cos (E) - e] + (1-e^2)^{1/2} \cos I \cos \omega \sin E\} \qquad (190)$$

$$\frac{\partial X_0}{\partial e} = a\{(\cos \Omega \cos \omega - \sin \Omega \sin \omega \cos I) [Ae]$$

$$- [Be] E (\cos \Omega \sin \omega + \sin \Omega \cos \omega \cos I)\} \qquad (191)$$

$$\frac{\partial Y_0}{\partial e} = a\{(\cos \Omega \cos \omega + \sin \Omega \sin \omega \cos I) [Ae]$$

$$- [Be] (\sin \Omega \sin \omega - \cos \Omega \cos \omega \cos I)\} \qquad (192)$$

$$\frac{\partial Z_0}{\partial e} = a \{\sin I \sin \omega [Ae] + [Be] \sin I \cos \omega\} \qquad (193)$$

where

$$[Ae] = -\left[\frac{\sin^2 E}{1 + 1 - e \cos E}\right]$$

$$[Be] = \frac{\sin E (\cos E - e)}{(1-e \cos E)(1-e^2)^{1/2}}$$

It is noted that the problem will be concerned with observed times t_i. Since M is a function of t_i, the time of epoch τ will be used as an independent variable.

$$\frac{\partial X^\circ}{\partial \eta} = \begin{cases} -a\,[(\cos\omega\cos\Omega - \sin\Omega\sin\omega\cos I)\sin(E) \\ +\,(1-e^2)^{1/2}(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos I)\cos E]\,\dfrac{t_i - \tau}{(1-e\cos E)} \\ -2\,R_x\,\mu^{1/3}\,\eta^{-5/3}/3a \end{cases} \tag{194}$$

$$\frac{\partial Y^\circ}{\partial \eta} = \begin{cases} -a\,[(\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos I)\sin(E) \\ +\,(1-e^2)^{1/2}(\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos I)\cos E]\,\dfrac{t_i - \tau}{(1-e\cos E)} \\ -2\,R_y\,\mu^{1/3}\,\eta^{-5/3}/3a \end{cases} \tag{195}$$

$$\frac{\partial Z^\circ}{\partial \eta} = \begin{cases} -a\,[\sin I\sin\omega\sin(E) - (1-e^2)^{1/2}\sin I\cos\omega\cos E]\,\dfrac{t_i - \tau}{(1-e\cos E)} \\ -2\,R_z\,\mu^{1/3}\,\eta^{-5/3}/3a \end{cases} \tag{196}$$

$$\frac{\partial X^\circ}{\partial \tau} = \frac{\partial X^\circ}{\partial \eta}\cdot\frac{-\eta}{t_i - \tau} \tag{197}$$

$$\frac{\partial Y^\circ}{\partial \tau} = \frac{\partial Y^\circ}{\partial \eta}\cdot\frac{-\eta}{t_i - \tau} \tag{198}$$

$$\frac{\partial Z^\circ}{\partial \tau} = \frac{\partial Z^\circ}{\partial \eta}\cdot\frac{-\eta}{t_i - \tau} \tag{199}$$

$$\frac{\partial F_{1i}}{\partial \omega_j} = \frac{+f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{11}}{\partial \omega_j} - \Gamma_{1i}\frac{\partial m_{31}}{\partial \omega_j}\right] + (Y_i - Y_0)\left[\frac{\partial m_{12}}{\partial \omega_j} - \Gamma_{1i}\frac{\partial m_{32}}{\omega_j}\right] + (Z_i - Z_0)\left[\frac{\partial m_{13}}{\partial \omega_j} - \Gamma_{1i}\frac{\partial m_{33}}{\partial \omega_j}\right]\right\} \tag{200}$$

$$\frac{\partial F_{1i}}{\partial \varphi} = \frac{+f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{11}}{\partial \varphi} - \Gamma_{1i}\frac{\partial m_{31}}{\partial \varphi}\right] + (Y_i - Y_0)\left[\frac{\partial m_{12}}{\partial \varphi} - \Gamma_{1i}\frac{\partial m_{32}}{\partial \varphi}\right] + (Z_i - Z_0)\left[\frac{\partial m_{13}}{\partial \varphi} - \Gamma_{1i}\frac{\partial m_{33}}{\partial \varphi}\right]\right\} \quad (201)$$

$$\frac{\partial F_{1i}}{\partial \kappa} = \frac{+f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{11}}{\partial \kappa} - \Gamma_{1i}\frac{\partial m_{31}}{\partial \kappa}\right] + (Y_i - Y_0)\left[\frac{\partial m_{12}}{\partial \kappa} - \Gamma_{1i}\frac{\partial m_{32}}{\partial \kappa}\right] + (Z_i - Z_0)\left[\frac{\partial m_{13}}{\partial \kappa} - \Gamma_{1i}\frac{\partial m_{33}}{\partial \varphi}\right]\right\} \quad (202)$$

$$\frac{\partial F_{2i}}{\partial \omega_j} = \frac{+f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{21}}{\partial \omega_j} - \Gamma_{2i}\frac{\partial m_{31}}{\partial \omega_j}\right] + (Y_i - Y_0)\left[\frac{\partial m_{22}}{\partial \omega_j} - \Gamma_{2i}\frac{\partial m_{32}}{\partial \omega_j}\right] + (Z_i - Z_0)\left[\frac{\partial m_{23}}{\partial \omega_j} - \Gamma_{2i}\frac{\partial m_{33}}{\partial \omega_j}\right]\right\} \quad (203)$$

$$\frac{\partial F_{2i}}{\partial \varphi} = \frac{+f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{21}}{\partial \varphi} - \Gamma_{2i}\frac{\partial m_{31}}{\partial \varphi}\right] + (Y_i - Y_0)\left[\frac{\partial m_{22}}{\partial \varphi} - \Gamma_{2i}\frac{\partial m_{32}}{\partial \varphi}\right] + (Z_i - Z_0)\left[\frac{\partial m_{23}}{\partial \varphi} - \Gamma_{2i}\frac{\partial m_{33}}{\partial \varphi}\right]\right\} \quad (204)$$

$$\frac{\partial F_{2i}}{\partial \kappa} = \frac{+f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{21}}{\partial \kappa} - \Gamma_{2i}\frac{\partial m_{31}}{\partial \kappa}\right] + (Y_i - Y_0)\left[\frac{\partial m_{22}}{\partial \kappa} - \Gamma_{2i}\frac{\partial m_{32}}{\partial \kappa}\right] + (Z_i - Z_0)\left[\frac{\partial m_{23}}{\partial \kappa} - \Gamma_{2i}\frac{\partial m_{33}}{\partial \kappa}\right]\right\} \quad (205)$$

or in general

$$\frac{\partial F_{ki}}{\partial \xi} = \frac{-f}{W_i}\left\{(X_i - X_0)\left[\frac{\partial m_{ki}}{\partial \xi} - \Gamma_{ki}\frac{\partial m_{31}}{\partial \xi}\right] + (Y_i - Y_0)\left[\frac{\partial m_{k2}}{\partial \xi} - \Gamma_{ki}\frac{\partial m_{32}}{\partial \xi}\right] + (Z_i - Z_0)\left[\frac{\partial m_{k3}}{\partial \xi} - \Gamma_{ki}\frac{\partial m_{33}}{\partial \xi}\right]\right\} \quad (206)$$

where

$$\Gamma_{ki} = \frac{m_{ki}(X_i - X_0) + m_{k2}(Y_i - Y_0) + m_{k3}(Z_i - Z_0)}{m_{31}(X_i - X_0) + m_{32}(Y_i - Y_0) + m_{33}(Z_i - Z_0)}$$

$k = 1, 2$

$$\frac{\partial m_{11}}{\partial \omega_j} = \cos \lambda_0 \left[(\sin \kappa \cos \omega_j + \cos \kappa \sin \varphi \sin \omega_j) \cos \Phi_0 - (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \sin A_0 \sin \Phi_0\right] + \sin \lambda_0 \cos A_0 (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \quad (207)$$

$$\frac{\partial m_{12}}{\partial \omega_j} = \sin \lambda_0 \left[(\sin \kappa \cos \omega + \cos \kappa \sin \varphi \sin \omega) \cos \Phi_0 - (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \sin A_0 \sin \Phi_0\right] - \cos \lambda_0 \cos A_0 (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \quad (208)$$

$$\frac{\partial m_{13}}{\partial \omega_j} = (\cos \kappa \sin \varphi \sin \omega + \sin \kappa \cos \omega) \sin \Phi_0 + (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \sin A_0 \cos \Phi_0 \quad (209)$$

$$\frac{\partial m_{21}}{\partial \omega_j} = \cos \lambda_0 \left[(\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \sin A_0 \sin \Phi_0 + (\cos \kappa \cos \omega - \sin \kappa \sin \varphi \sin \omega) \cos \Phi_0\right] - \sin \lambda_0 \cos A_0 (\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \quad (210)$$

$$\frac{\partial m_{22}}{\partial \omega_j} = \sin \lambda_0 \left[(\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \sin A_0 \sin \Phi_0 + (\cos \kappa \cos \omega - \sin \kappa \sin \varphi \sin \omega) \cos \Phi_0\right] + \cos \lambda_0 (\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \quad (211)$$

$$\frac{\partial m_{23}}{\partial \omega_j} = (\cos \kappa \cos \omega - \sin \kappa \sin \varphi \sin \omega) \sin \Phi_0 - (\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \sin A_0 \cos \Phi_0 \quad (212)$$

$$\frac{\partial m_{31}}{\partial \omega_j} = \cos \lambda_0 \left[\cos \varphi \cos \omega \sin A_0 \sin \Phi_0 - \cos \varphi \sin \omega \cos \Phi_0\right] - \sin \lambda_0 \cos \varphi \cos A_0 \tag{213}$$

$$\frac{\partial m_{32}}{\partial \omega_j} = \sin \lambda_0 \left[\cos \varphi \cos \omega \sin A_0 \sin \Phi_0 - \cos \varphi \sin \omega \cos \Phi_0\right] + \cos \lambda_0 \cos \varphi \cos \omega \cos A_0 \tag{214}$$

$$\frac{\partial m_{33}}{\partial \omega_j} = - \cos \varphi \cos \omega \sin A_0 \cos \Phi_0 - \cos \varphi \sin \omega \sin \Phi_0 \tag{215}$$

$$\frac{\partial m_{11}}{\partial \varphi} = \cos \lambda_0 \left\{\left[\cos \kappa \sin \varphi \cos A_0 - \cos \kappa \cos \varphi \sin \omega \sin A_0\right] \sin \Phi_0 - \cos \kappa \cos \varphi \cos \omega \cos \Phi_0\right\} + \sin \lambda_0 \left[\cos \kappa \sin \varphi \sin A_0 + \cos \kappa \cos \varphi \sin \omega \cos A_0\right] \tag{216}$$

$$\frac{\partial m_{12}}{\partial \varphi} = \sin \lambda_0 \left\{\left[\cos \kappa \sin \varphi \cos A_0 - \cos \kappa \cos \varphi \sin \omega \sin A_0\right] \sin \Phi_0 - \cos \kappa \cos \varphi \cos \omega \cos \Phi_0\right\} - \cos \lambda_0 \left[\cos \kappa \sin \varphi \sin A_0 + \cos \kappa \cos \varphi \sin \omega \cos A_0\right] \tag{217}$$

$$\frac{\partial m_{13}}{\partial \varphi} = \cos \Phi_0 \left(\cos \kappa \cos \varphi \sin \omega \sin A_0 - \cos \kappa \sin \varphi \cos A_0\right) - \cos \kappa \cos \varphi \cos \omega \sin \Phi_0 \tag{218}$$

$$\frac{\partial m_{21}}{\partial \varphi} = \cos \lambda_0 \left\{\left[\sin \kappa \cos \varphi \cos \omega \cos \Phi_0 - \left[\sin \kappa \sin \varphi \cos A_0 + \sin \kappa \cos \varphi \sin \omega \sin A_0\right] \sin \Phi_0\right\} - \sin \lambda_0 \left[\sin \kappa \sin \varphi \sin A_0 + \sin \kappa \cos \varphi \sin \omega \cos A_0\right] \tag{219}$$

$$\frac{\partial m_{22}}{\partial \varphi} = \sin \lambda_0 \left[(\sin \kappa \sin \omega \cos \varphi \sin A_0 - \sin \kappa \sin \varphi \cos A_0) \sin \Phi_0 \right.$$

$$\left. + \sin \kappa \cos \varphi \cos \omega \cos \Phi_0\right]$$

$$+ \cos \lambda_0 (\sin \kappa \sin \varphi \sin A_0 + \sin \kappa \cos \varphi \sin \omega \cos A_0) \qquad (220)$$

$$\frac{\partial m_{23}}{\partial \varphi} = (\sin \kappa \sin \varphi \cos A_0 - \sin \kappa \cos \varphi \sin \omega \sin A_0) \cos \Phi_0$$

$$+ \sin \kappa \cos \varphi \cos \omega \sin \Phi_0 \qquad (221)$$

$$\frac{\partial m_{31}}{\partial \varphi} = \sin \lambda_0 (\sin \varphi \sin \omega \cos A_0 - \cos \varphi \sin A_0)$$

$$- \cos \lambda_0 \left[(\cos \varphi \cos A_0 + \sin \varphi \sin \omega \sin A_0) \sin \Phi_0 \right.$$

$$\left. + \sin \varphi \cos \omega \cos \Phi_0\right] \qquad (222)$$

$$\frac{\partial m_{32}}{\partial \varphi} = \cos \lambda_0 (\cos \varphi \sin A_0 - \sin \varphi \sin \omega \cos A_0)$$

$$- \sin \lambda_0 \left[(\cos \varphi \cos A_0 + \sin \varphi \sin \omega \sin A_0) \sin \Phi_0 \right.$$

$$\left. + \sin \varphi \cos \omega \cos \Phi_0\right] \qquad (223)$$

$$\frac{\partial m_{33}}{\partial \varphi} = (\cos \varphi \cos A_0 + \sin \varphi \sin \omega \sin A_0) \cos \Phi_0$$

$$- \sin \varphi \cos \omega \sin \Phi_0 \qquad (224)$$

$$\frac{\partial m_{11}}{\partial \kappa} = \cos \lambda_0 \{[\sin \kappa \cos \varphi \cos A_0 + (\sin \kappa \sin \varphi \sin \omega - \cos \kappa \cos \omega) \sin A_0]$$

$$\sin \Phi_0 + (\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \cos \Phi_0\}$$

$$+ \sin \lambda_0 [\sin \kappa \cos \varphi \sin A_0 + (\cos \kappa \cos \omega - \sin \kappa \sin \varphi \sin \omega) \cos A_0]$$

$$(225)$$

$$\frac{\partial m_{12}}{\partial \kappa} = \sin \lambda_0 \{[\sin \kappa \cos \varphi \cos A_0 + (\sin \kappa \sin \varphi \sin \omega - \cos \kappa \cos \omega) \sin A_0]$$

$$\sin \Phi_0 + (\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega) \cos \Phi_0\}$$

$$- \cos \lambda_0 [\sin \kappa \cos \varphi \sin A_0 + (\cos \kappa \cos \omega - \sin \kappa \sin \varphi \sin \omega) \cos A_0] \tag{226}$$

$$\frac{\partial m_{13}}{\partial \kappa} = [(\cos \kappa \cos \omega - \sin \kappa \sin \varphi \sin \omega) \sin A_0 - \sin \kappa \cos \varphi \cos A_0] \cos \Phi_0$$

$$+ [\cos \kappa \sin \omega + \sin \kappa \sin \varphi \cos \omega] \sin \Phi_0 \tag{227}$$

$$\frac{\partial m_{21}}{\partial \kappa} = \cos \lambda_0 \{[(\cos \kappa \cos \varphi \cos A_0) + (\sin \kappa \cos \omega + \cos \kappa \sin \varphi \sin \omega) \sin A_0]$$

$$\sin \Phi_0 + [\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega] \cos \Phi_0\}$$

$$+ \sin \lambda_0 [\cos \kappa \cos \varphi \sin A_0 - (\sin \kappa \cos \omega + \cos \kappa \sin \varphi \sin \omega) \cos A_0] \tag{228}$$

$$\frac{\partial m_{22}}{\partial \kappa} = \sin \lambda_0 \{[\cos \kappa \cos \varphi \cos A_0 + (\sin \kappa \cos \omega + \cos \kappa \sin \varphi \sin \omega) \sin A_0]$$

$$\sin \Phi_0 + (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \cos \Phi_0\}$$

$$- \cos \lambda_0 [\cos \kappa \cos \varphi \sin A_0 - (\sin \kappa \cos \omega + \cos \kappa \sin \varphi \sin \omega) \cos A_0] \tag{229}$$

$$\frac{\partial m_{23}}{\partial \kappa} = \cos \Phi_0 [-\cos \kappa \cos \varphi \cos A_0 - (\sin \kappa \cos \omega + \cos \kappa \sin \varphi \sin \omega) \sin A_0]$$

$$+ (\cos \kappa \sin \varphi \cos \omega - \sin \kappa \sin \omega) \sin \Phi_0 \tag{230}$$

$$\frac{\partial m_{31}}{\partial \kappa} = 0 \tag{231}$$

$$\frac{\partial m_{32}}{\partial \kappa} = 0 \tag{232}$$

$$\frac{\partial m_{33}}{\partial \kappa} = 0 \tag{233}$$

$$\frac{\partial m_{11}}{\partial\Omega} = \{-[\cos\kappa\cos\varphi\cos A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\sin A_0]\sin\Phi_0$$

$$+ (\sin\kappa\sin\omega_j - \cos\kappa\sin\varphi\cos\omega_j)\cos\Phi_0\}\,\partial(\cos\lambda_0)/\partial\Omega$$

$$+ \{-\cos\kappa\cos\varphi\sin A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\cos A_0\} \times$$

$$\partial(\sin\lambda_0)/\partial\Omega$$

$$+ \{-\cos\lambda_0[\cos\kappa\cos\varphi\cos A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)$$

$$\sin A_0]\} \times \partial(\sin\Phi_0)/\partial\Omega$$

$$+ [\cos\lambda_0(\sin\kappa\sin\omega_j - \cos\kappa\sin\varphi\cos\omega_j)]\,\partial(\cos\Phi_0)/\partial\Omega$$

$$+ \{-\cos\lambda_0[\cos\kappa\cos\varphi\sin\Phi_0 + \sin\lambda_0$$

$$(\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)]\} \times \partial(\cos A_0)/\partial\Omega$$

$$+ \{-\cos\lambda_0[(\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)]$$

$$\sin\Phi_0 - \sin\lambda_0\cos\varphi\cos\kappa\} \times \partial(\sin A_0)/\partial\Omega \qquad (234)$$

$$\frac{\partial m_{12}}{\partial\Omega} = \{\cos\kappa\cos\varphi\sin A_0 - (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\cos A_0\} \times$$

$$\partial(\cos\lambda_0)/\partial\Omega$$

$$+ \{-[\cos\kappa\cos\varphi\cos A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\sin A_0]$$

$$\sin\Phi_0 + (\sin\kappa\sin\omega_j - \cos\kappa\sin\varphi\cos\omega_j)\cos\Phi_0\}\,\partial(\sin\lambda_0)/\partial\Omega$$

$$+ \{-\sin\lambda_0[\cos\kappa\cos\varphi\cos A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)$$

$$\sin A_0]\} \times \partial(\sin\Phi_0)/\partial\Omega + \{\sin\lambda_0(\sin\kappa\sin\omega_j - \cos\kappa\sin\varphi\cos\omega_j)\}$$

$$\partial(\cos\Phi_0)/\partial\Omega$$

$$+ \{-\sin\lambda_0\cos\kappa\cos\varphi\sin\Phi_0 - \cos\lambda_0(\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\}$$

$$\times\,\partial(\cos A_0)/\partial\Omega + \{-\sin\lambda_0\sin\Phi_0(\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)$$

$$+ \cos\lambda_0\cos\kappa\cos\varphi\}\,\partial(\sin A_0)/\partial\Omega \qquad (235)$$

$$\frac{\partial m_{13}}{\partial\Omega} = [\cos\kappa\cos\varphi\cos A_0 + (\sin\kappa\cos\omega_j + \cos\kappa\sin\varphi\sin\omega_j)\sin A_0]$$

$$\partial(\cos\Phi_0)/\partial\Omega$$

$$+ [\sin\kappa\sin\omega_j - \cos\kappa\sin\varphi\cos\omega_j]\,\partial(\sin\Phi_0)/\partial\Omega + [\sin\kappa\cos\omega_j$$

$$+ \cos\kappa\sin\varphi\sin\omega_j]\cos\Phi_0 \cdot \partial(\sin A_0)/\partial\Omega + [\cos\kappa\cos\varphi\cos\Phi_0]$$

$$\partial(\cos A_0)/\partial\Omega \qquad (236)$$

$$
\begin{aligned}
\frac{\partial m_{21}}{\partial \Omega} = & \{(\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j) \cos\Phi_0 + [\sin\kappa \cos\varphi \cos A_0 \\
& - (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \sin A_0] \sin\Phi_0\} \, \partial(\cos\lambda_0)/\partial\Omega \\
& + [\sin\kappa \cos\varphi \sin A_0 + (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \cos A_0] \\
& \partial(\sin\lambda_0)/\partial\Omega \\
& + \{\cos\lambda_0 [\sin\kappa \cos\varphi \cos A_0 - (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \\
& \sin A_0]\} \, \partial(\sin\Phi_0)/\partial\Omega + \{\cos\lambda_0 [\sin\kappa \cos\varphi \sin\Phi_0] \\
& + \sin\lambda_0 (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j)\} \, \partial(\cos A_0)/\partial\Omega \\
& + [\sin\lambda_0 \sin\kappa \cos\varphi - \cos\lambda_0 (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \\
& \sin\Phi_0] \, \partial(\sin A_0)/\partial\Omega \\
& + \cos\lambda_0 (\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j) \, \partial(\cos\Phi_0)/\partial\Omega \qquad (237)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial m_{22}}{\partial \Omega} = & \{(\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j) \cos\Phi_0 + [\sin\kappa \cos\varphi \cos A_0 \\
& - (\cos\kappa \cos\omega_j - \sin\kappa \sin\omega_j \sin\varphi) \sin A_0] \sin\Phi_0\} \, \partial(\sin\lambda_0)/\partial\Omega \\
& - [\sin\kappa \cos\varphi \sin A_0 + (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \cos A_0] \partial(\cos\lambda_0)/\partial\Omega \\
& + [\sin\lambda_0 (\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j)] \, \partial(\cos\Phi_0)/\partial\Omega \\
& + \sin\lambda_0 [\sin\kappa \cos\varphi \cos A_0 - (\cos\kappa \cos\omega - \sin\kappa \sin\omega \sin\varphi) \sin A_0] \\
& \partial(\sin\Phi_0)/\partial\Omega + [\sin\lambda_0 \sin\kappa \cos\varphi \sin\Phi_0 - \cos\lambda_0 \\
& (\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j)] \, \partial(\cos A_0)/\partial\Omega \\
& - [\sin\lambda_0 (\cos\kappa \cos\omega_j - \sin\kappa \sin\omega_j \sin\varphi) \sin\Phi_0 \\
& + \cos\lambda_0 \sin\kappa \cos\varphi] \, \partial(\sin A_0)/\partial\Omega \qquad (238)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial m_{23}}{\partial \Omega} = & [(\cos\kappa \cos\omega_j - \sin\kappa \sin\varphi \sin\omega_j) \sin A_0 - \sin\kappa \cos\varphi \cos A_0] \\
& \partial(\cos\Phi_0)/\partial\Omega + [\cos\kappa \sin\omega_j + \sin\kappa \sin\varphi \cos\omega_j] \, \partial(\sin\Phi_0)/\partial\Omega \\
& - [\cos\Phi_0 \sin\kappa \cos\varphi] \, \partial(\cos A_0)/\partial\Omega + [(\cos\kappa \cos\omega_j \\
& - \sin\kappa \sin\varphi \sin\omega_j) \cos\Phi_0] \, \partial(\sin A_0)/\partial\Omega \qquad (239)
\end{aligned}
$$

$$\frac{\partial m_{31}}{\partial \Omega} = [\cos\varphi \cos\omega_j \cos\Phi_0 - (\sin\varphi \cos A_0 - \cos\varphi \sin\omega_j \sin A_0) \sin\Phi_0] \partial(\cos\lambda_0)/\partial\Omega - [\sin\varphi \sin A_0 + \cos\varphi \sin\omega_j \cos A_0] \partial(\sin\lambda_0)/\partial\Omega + [\cos\lambda_0 \cos\varphi \cos\omega_j] \partial(\cos\Phi_0)/\partial\Omega - \{\cos\lambda_0 [\sin\varphi \cos A_0 - \cos\varphi \sin\omega_j \sin A_0]\} \partial(\sin\Phi_0)/\partial\Omega - [\sin\lambda_0 \cos\varphi \sin\omega_j + \sin\varphi \cos\lambda_0 \sin\Phi_0] \partial(\cos A_0)/\partial\Omega + [\cos\lambda_0 \cos\varphi \sin\omega_j \sin\Phi_0 - \sin\lambda_0 \sin\varphi] \partial(\sin A_0)/\partial\Omega \tag{240}$$

$$\frac{\partial m_{32}}{\partial \Omega} = [\sin\varphi \sin A_0 + \cos\varphi \sin\omega_j \cos A_0] \partial(\cos\lambda_0)/\partial\Omega + \{\cos\varphi \cos\omega_j \cos\Phi_0 - [\sin\varphi \cos A_0 - \cos\varphi \sin\omega_j \sin A_0)] \sin\Phi_0\} \partial(\sin\lambda_0)/\partial\Omega + [\sin\lambda_0 \cos\varphi \cos\omega_j] \partial(\cos\Phi_0)/\partial\Omega - [\sin\lambda_0 (\sin\varphi \cos A_0 - \cos\varphi \sin\omega_j \sin A_0)] \partial(\sin\Phi_0)/\partial\Omega + [\cos\lambda_0 \cos\varphi \sin\omega_j - \sin\lambda_0 \sin\varphi \sin\Phi_0] \partial(\cos A_0)/\partial\Omega + [\sin\lambda_0 \cos\varphi \sin\omega_j \sin\Phi_0 + \cos\lambda_0 \sin\varphi)] \partial(\sin A_0)/\partial\Omega \tag{241}$$

$$\frac{\partial m_{33}}{\partial \Omega} = [(\sin\varphi \cos A_0 - \cos\varphi \sin\omega_j \sin A_0)] \partial(\cos\Phi_0)/\partial\Omega + [\cos\varphi \cos\omega_j] \partial(\sin\Phi_0)/\partial\Omega + [\sin\varphi \cos\Phi_0] \partial(\cos A_0)/\partial\Omega - [\cos\varphi \sin\omega_j \cos\Phi_0] \partial(\sin A_0)/\partial\Omega \tag{242}$$

$$\frac{\partial(\cos\lambda_0)}{\partial\Omega} = \frac{Y}{(X^2 + Y^2)^{3/2}} \left[Y \frac{\partial X}{\partial\Omega} - X \frac{\partial Y}{\partial\Omega}\right] \tag{243}$$

$$\frac{\partial(\sin\lambda_0)}{\partial\Omega} = -\frac{X}{Y} \cdot \frac{(\cos\lambda_0)}{\partial\Omega} \tag{244}$$

$$\frac{\partial(\cos\Phi_0)}{\partial\Omega} = \frac{\sin\Phi_0 \tan\Phi_0}{R^2} \left[X \frac{\partial X}{\partial\Omega} + Y \frac{\partial Y}{\partial\Omega}\right] \tag{245}$$

$$\frac{\partial(\sin\Phi_0)}{\partial\Omega} = -\cot\Phi_0 \frac{\partial(\cos\Phi_0)}{\partial\Omega} \tag{246}$$

$$\frac{\partial\,(\cos A_0)}{\partial\Omega} = \frac{\sin I}{Y}\left[\sin\,(\lambda - \Omega) + \frac{X \sin\Omega + Y \cos\Omega}{Y}\;\frac{\partial\,(\cos\lambda_0)}{\partial\Omega}\right] \tag{247}$$

$$\frac{\partial\,(\sin A_0)}{\partial\Omega} = -\,\frac{\sin A}{\cos\Phi}\;\frac{\partial\,(\cos\Phi)}{\partial\Omega} \tag{248}$$

$$\frac{\partial\,(\cos\lambda_0)}{\partial\omega} = \frac{Y}{(X^2 + Y^2)^{3/2}}\left[Y\,\frac{\partial X}{\partial\omega} - X\,\frac{\partial Y}{\partial\omega}\right] \tag{249}$$

$$\frac{\partial\,(\sin\lambda_0)}{\partial\omega} = -\,\frac{X}{Y}\cdot\frac{\partial\,(\cos\lambda_0)}{\partial\omega} \tag{250}$$

$$\frac{\partial\,(\cos\Phi_0)}{\partial\omega} = \frac{1}{R^2}\left\{\sin\Phi_0\,\tan\Phi_0\left[X\,\frac{\partial X}{\partial\omega} + Y\,\frac{\partial Y}{\partial\omega}\right] - Z\,\cos\Phi_0\,\frac{\partial Z}{\partial\omega}\right\} \tag{251}$$

$$\frac{\partial\,(\sin\Phi_0)}{\partial\omega} = \frac{1}{R}\left\{\cos^2\Phi_0\,\frac{\partial Z}{\partial\omega} - \frac{\sin\Phi_0}{R}\left[X\,\frac{\partial X}{\partial\omega} + Y\,\frac{\partial Y}{\partial\omega}\right]\right\} \tag{252}$$

$$\frac{\partial\,(\cos A_0)}{\partial\omega} = \frac{\sin I\,(X \sin\Omega + Y \cos\Omega)}{Y}\cdot\frac{\partial\,(\cos\lambda_0)}{\partial\omega} \tag{253}$$

$$\frac{\partial\,(\sin A_0)}{\partial\omega} = -\,\frac{\sin A_0}{\cos\Phi_0}\;\frac{\partial\,(\cos\Phi_0)}{\partial\omega} \tag{254}$$

$$\frac{\partial\,(\cos\lambda_0)}{\partial I} = \frac{Y}{(X^2 + Y^2)^{3/2}}\left[Y\,\frac{\partial X}{\partial I} - X\,\frac{\partial Y}{\partial I}\right] \tag{255}$$

$$\frac{\partial\,(\sin\lambda_0)}{\partial I} = -\,\frac{X}{Y}\cdot\frac{\partial\,(\cos\lambda_0)}{\partial I} \tag{256}$$

$$\frac{\partial\,(\cos\Phi_0)}{\partial I} = \frac{1}{R^2}\left\{\sin\Phi_0\,\tan\Phi_0\left[X\,\frac{\partial X}{\partial I} + Y\,\frac{\partial Y}{\partial I}\right] - Z\,\cos\Phi_0\,\frac{\partial Z}{\partial I}\right\} \tag{257}$$

$$\frac{\partial\,(\sin\Phi_0)}{\partial I} = \frac{1}{R}\left\{\cos^2\Phi\,\frac{\partial Z}{\partial I} - \frac{\sin\Phi_0}{R}\left[X\,\frac{\partial X}{\partial I} + Y\,\frac{\partial Y}{\partial I}\right]\right\} \tag{258}$$

$$\frac{\partial (\cos A_0)}{\partial I} = \cos I \cos(\lambda_0 - \Omega) + \frac{\sin I}{Y} (X \sin \Omega + Y \cos \Omega) \frac{\partial (\cos \lambda_0)}{\partial I} \tag{259}$$

$$\frac{\partial (\sin A_0)}{\partial I} = - \frac{1}{\cos \Phi_0} \left[\sin I + \sin A_0 \cdot \frac{\partial (\cos \Phi_0)}{\partial I} \right] \tag{260}$$

$$\frac{\partial (\cos \lambda_0)}{\partial e} = \frac{Y}{(X^2 + Y^2)^{3/2}} \left[Y \frac{\partial X}{\partial e} - X \frac{\partial Y}{\partial e} \right] \tag{261}$$

$$\frac{\partial (\sin \lambda_0)}{\partial e} = - \frac{X}{Y} \cdot \frac{\partial (\cos \lambda_0)}{\partial e} \tag{262}$$

$$\frac{\partial (\cos \Phi_0)}{\partial e} = \frac{1}{R^2} \left\{ \sin \Phi_0 \tan \Phi_0 \left[X \frac{\partial X}{\partial e} + Y \frac{\partial Y}{\partial e} \right] - Z \cos \Phi_0 \frac{\partial Z}{\partial e} \right\} \tag{263}$$

$$\frac{\partial (\sin \Phi_0)}{\partial e} = \frac{1}{R} \left\{ \cos^2 \Phi \frac{\partial Z}{\partial e} - \frac{\sin \Phi}{R} \left[X \frac{\partial X}{\partial e} + Y \frac{\partial Y}{\partial e} \right] \right\} \tag{264}$$

$$\frac{\partial (\cos A_0)}{\partial e} = \sin I \; \frac{[X \sin \Omega + Y \cos \Omega]}{Y} \frac{\partial (\cos \lambda_0)}{\partial e} \tag{265}$$

$$\frac{\partial (\sin A_0)}{\partial e} = - \frac{\sin A_0}{\cos \Phi_0} \cdot \frac{\partial (\cos \Phi_0)}{\partial e} \tag{266}$$

$$\frac{\partial (\cos \lambda_0)}{\partial \eta} = \frac{Y}{(X^2 + Y^2)^{3/2}} \left[Y \frac{\partial X}{\partial \eta} - X \frac{\partial Y}{\partial \eta} \right] \tag{267}$$

$$\frac{\partial (\sin \lambda_0)}{\partial \eta} = - \frac{X}{Y} \cdot \frac{\partial (\cos \lambda_0)}{\partial \eta} \tag{268}$$

$$\frac{\partial (\cos \Phi_0)}{\partial \eta} = \frac{1}{R^2} \left\{ \sin \Phi_0 \tan \Phi_0 \left[X \frac{\partial X}{\partial \eta} + Y \frac{\partial Y}{\partial \eta} \right] - Z \cos \Phi_0 \frac{\partial Z}{\partial \eta} \right\} \tag{269}$$

$$\frac{\partial (\sin \Phi_0)}{\partial \eta} = \frac{1}{R} \left\{ \cos^2 \Phi_0 \frac{\partial Z}{\partial \eta} - \frac{\sin \Phi_0}{R} \left[X \frac{\partial X}{\partial \eta} + Y \frac{\partial Y}{\partial \eta} \right] \right\} \tag{270}$$

$$\frac{\partial (\cos A_0)}{\partial \eta} = \sin I \frac{[X \sin \Omega + Y \cos \Omega]}{Y} \frac{\partial (\cos \lambda_0)}{\partial \eta} \tag{271}$$

$$\frac{\partial (\sin A_0)}{\partial \eta} = - \frac{\sin A_0}{\cos \Phi_0} \cdot \frac{\partial (\cos \Phi_0)}{\partial \eta} \tag{272}$$

$$\frac{\partial (\cos \lambda_0)}{\partial \tau} = \frac{Y}{(X^2 + Y^2)^{3/2}} \left[Y \frac{\partial X}{\partial \tau} - X \frac{\partial Y}{\partial \tau} \right] \tag{273}$$

$$\frac{\partial (\sin \lambda_0)}{\partial \tau} = - \frac{X}{Y} \frac{\partial (\cos \lambda_0)}{\partial \tau} \tag{274}$$

$$\frac{\partial (\cos \Phi_0)}{\partial \tau} = \frac{1}{R^2} \left\{ \sin \Phi_0 \tan \Phi_0 \left[X \frac{\partial X}{\partial \tau} + Y \frac{\partial Y}{\partial \tau} \right] - Z \cos \Phi_0 \frac{\partial Z}{\partial \tau} \right\} \tag{275}$$

$$\frac{\partial (\sin \Phi_0)}{\partial \tau} = \frac{1}{R} \left\{ \cos^2 \Phi \frac{\partial Z}{\partial \tau} - \frac{\sin \Phi}{R} \left[X \frac{\partial X}{\partial \tau} + Y \frac{\partial Y}{\partial \tau} \right] \right\} \tag{276}$$

$$\frac{\partial (\cos A_0)}{\partial \tau} = \sin I \frac{[X \sin \Omega + Y \cos \Omega]}{Y} \frac{\partial (\cos \lambda_0)}{\partial \tau} \tag{277}$$

$$\frac{\partial (\sin A_0)}{\partial \tau} = - \frac{\sin A_0}{\cos \Phi_0} \cdot \frac{\partial (\cos \Phi_0)}{\partial \tau} \tag{278}$$

Now prepared to form the partial derivations of F_1, F_2 with respect to the orbital elements. For any element ξ, may write

$$\frac{\partial F_1}{\partial \xi} = \frac{+f}{W} \left[\frac{\partial U}{\partial \xi} - \frac{U}{W} \frac{\partial W}{\partial \xi} \right] \tag{279}$$

and

$$\frac{\partial F_2}{\partial \xi} = \frac{+f}{W} \left[\frac{\partial V}{\partial \xi} - \frac{V}{W} \frac{\partial W}{\partial \xi} \right] \tag{280}$$

which became

$$\frac{\partial F_1}{\partial \xi} = \frac{+f}{W}\left\{(X_i - X_0)\left[\frac{\partial m_{11}}{\partial \xi} - \frac{U}{W}\frac{\partial m_{31}}{\partial \xi}\right] + (Y_i - Y_0)\left[\frac{\partial m_{12}}{\partial \xi} - \frac{U}{W}\frac{\partial m_{32}}{\partial \xi}\right]\right.$$
$$+ (Z_i - Z_0)\left[\frac{\partial m_{23}}{\partial \xi} - \frac{U}{W}\frac{\partial m_{33}}{\partial \xi}\right] + \left[\frac{U}{W} m_{31} - m_{11}\right]\frac{\partial X_0}{\partial \xi}$$
$$\left. + \left[\frac{U}{W} m_{32} - m_{12}\right]\frac{\partial Y_0}{\partial \xi} + \left[\frac{U}{W} m_{33} - m_{13}\right]\frac{\partial Z_0}{\partial \xi}\right\} \tag{281}$$

and

$$\frac{\partial F_2}{\partial \xi} = \frac{+f}{W}\left\{(X_i - X_0)\left[\frac{\partial m_{21}}{\partial \xi} - \frac{V}{W}\frac{\partial m_{31}}{\partial \xi}\right] + (Y_i - Y_0)\left[\frac{\partial m_{22}}{\partial \xi} - \frac{V}{W}\frac{\partial m_{32}}{\partial \xi}\right]\right.$$
$$+ (Z_i - Z_0)\left[\frac{\partial m_{23}}{\partial \xi} - \frac{V}{W}\frac{\partial m_{33}}{\partial \xi}\right] + \left[\frac{V}{W} m_{31} - m_{21}\right]\frac{\partial X_0}{\partial \xi}$$
$$\left. + \left[\frac{V}{W} m_{32} - m_{22}\right]\frac{\partial Y_0}{\partial \xi} + \left[\frac{V}{W} m_{33} - m_{23}\right]\frac{\partial Z_0}{\partial \xi}\right\} \tag{282}$$

generalized to

$$\frac{\partial F_{ki}}{\partial \xi} = \frac{+f}{W}\left\{(X_i - X_0)\left[\frac{\partial m_{k1}}{\partial \xi} - \Gamma_{ki}\frac{\partial m_{31}}{\partial \xi}\right] + (Y_i - Y_0)\left[\frac{\partial m_{k2}}{\partial \xi} - \Gamma_{ki}\frac{\partial m_{32}}{\partial \xi}\right]\right.$$
$$+ (Z_i - Z_0)\left[\frac{\partial m_{k3}}{\partial \xi} - \Gamma_{ki}\frac{\partial m_{33}}{\partial \xi}\right] + (\Gamma_{ki} m_{31} - m_{ki})\frac{\partial X_0}{\partial \xi}$$
$$\left. + (\Gamma_{ki} m_{32} - m_{k2})\frac{\partial Y_0}{\partial \xi} + (\Gamma_{ki} m_{33} - m_{k3})\frac{\partial Z_0}{\partial \xi}\right\} \tag{283}$$

where Γ_{ki} has been defined previously

Now all terms have been defined → coefficient matrices can now be formed.

[redacted]

Appendix C

INVERSION OF LARGE MATRICES

1.0 INTRODUCTION

One of the stumbling blocks preventing the rapid adoption of analytical photogrammetry is the problem of solving a large system of linear equations, which is intimately related to the problem of inverting a large matrix.

Consider a system of n linear equations relating n unknown quantities. Let the system be represented as

$$AX = R \tag{284}$$

where A signifies the coefficient matrix
R signifies the vector of constants
X signifies the vector of unknowns

Provided that the matrix A is not singular, the determination of the inverse matrix A^{-1} and of the individual unknowns x_i, may be readily obtained in theory. However, if A is a matrix of high order, a formidable amount of computing is required to obtain the solution. In general, one requires not only a specific solution for the vector X, but also a theoretical error analysis of the results. For this, it is necessary to determine the inverse matrix A^{-1}.

With the available high speed digital computers the problem is essentially one concerning the storage requirements for the quantities involved in the computations, the time taken to perform the calculations, and the precision of the results. This note will be an attempt to determine the optimum procedure which will furnish the required solution for X and for A^{-1}.

The methods by which the inverse of a matrix may be determined are defined as direct if the inverse may be determined by a finite number of operations, and as iterative if the inverse is determined as the limit of successive approximations. The direct solutions do not yield an exact solution, on account of the accumulated round-off errors. These errors may be considerable in the case of a large matrix. It will be necessary to determine the accuracy of the determination of A^{-1}, and to provide a method of correcting the calculated elements. In view of this, only the direct methods of evaluating the inverse will be considered, and a method of correcting the inexact solution will be given.

2.0 HOTELLING'S METHOD OF CORRECTING AN APPROXIMATE INVERSE MATRIX

Let the approximate inverse of the matrix A be D°, and define

$$F^\circ = I - AD^\circ \tag{285}$$

Note that if

$$D^\circ = A^{-1}$$

then

$$F^\circ = 0$$

Now consider the sequence of matrices

$$D_i = D_{i-1} (I + F_{i-1}) \tag{286}$$

and

$$F_i = (I - AD_i) \tag{287}$$

where $i = 1, 2, 3, \ldots, m$

so that

$$F_m = I - AD_m = F^2_{m-1} = F^4_{m-2} = \ldots = F^{\circ 2m} \tag{288}$$

Provided that the norm of F° is less than or equal to unity, the sequential determinations of D_1 converges rapidly. This is based on the convergence theorems:

If a system of linear equations is written in the form $X = AX^\circ + B$, the covergence of the iterative solution for the vector X starting with an initial vector X° and with any value of B is guaranteed if the proper numbers of the coefficient matrix A are less than unity in modulus.

From this theorem it follows that:

In order for the iterative process to converge, it is sufficient that the norm of the matrix A be less than unity.

3.0 MATRIX INVERSION BY THE SINGLE DIVISION SCHEME

This method is based on the theorem which states that if the leading submatrices of the square matrix A are non-zero, then A may be factored into the produce of a lower triangular matrix by an upper triangular matrix.

Consider the matrix A and the upper triangular matrix B and the lower triangular matrix C, such that A = CB.

This matrix C may be formed by using the recurrence formula

$$c_{ij} = a_{ij} - \sum_{k=1}^{j-1} (c_{ik} b_{kj}),\ i \geq j,\ c_{ik} = a_{ik} \tag{289}$$

and the matrix B by the recurrence formula

$$b_{ij} = \frac{a_{ij} - \sum_{c=1}^{i-1} (c_{ik} b_{kj})}{a_{ii}},\ i\ j,\ b_{ii} = 1 \tag{290}$$

Since $A^{-1} = B^{-1} C^{-1}$ then $A^{-1} C = B^{-1}$, in which B^{-1} is an upper triangular matrix with 1 along the leading diagonal. Similarly, $BA^{-1} = C^{-1}$. The equation for B^{-1} furnishes $n(n+1)/2$ equations of the form:

$$\left.\begin{aligned}
c_{11} a_{i1}^{-1} + c_{21} a_{i2}^{-1} + c_{31} a_{i3}^{-1} + \ldots + c_{n1} a_{in}^{-1} &= 1 \\
c_{22} a_{i2}^{-1} + c_{32} a_{i3}^{-1} + \ldots + c_{n2} a_{in}^{-1} &= 1 \\
c_{33} a_{i3}^{-1} + \ldots + c_{n3} a_{in}^{-1} &= 1 \\
&\vdots \\
c_{nn} a_{in}^{-1} &= 1
\end{aligned}\right\} \tag{291}$$

$$i = 1 \ldots n$$

Similarly, the equation for C^{-1} furnishes $n(n-1)/2$ equations of the form:

$$\left.\begin{aligned} a^{-1}_{1j} + b_{12}\, a^{-1}_{2j} + b_{13}\, a^{-1}_{3j} + \dots + b_{1n}\, a^{-1}_{nj} &= 0 \\ a^{-1}_{2j} + b_{23}\, a^{-1}_{3j} + \dots + b_{2n}\, a^{-1}_{nj} &= 0 \\ a^{-1}_{3j} + \dots + b_{3n}\, a^{-1}_{nj} &= 0 \\ &\;\vdots \\ a^{-1}_{n-1,j} + b_{(n-1)n}\, a^{-1}_{nj} &= 0 \end{aligned}\right\} \qquad j = 2 \dots n \tag{292}$$

From these two sets of equations the required values of a^{-1}_{ij} are determined, by putting $i = n$ in Equation (291) and solving for $a^{-1}_{nn} \dots a^{-1}_{n1}$, which are then substituted into Equation (292) to obtain $a^{-1}_{(n-1)n}$ for $j = n$.

A symmetric matrix S may be factored into two triangular matrices such that the lower triangular matrix is the transpose of the upper triangular matrix, i.e.,

$$S = D^T D \tag{293}$$

Then

$$S^{-1} = D^{-1}\, D^{T\,-1} \tag{294}$$

According to Equation (290)

$$d_{ij} = \frac{S_{ij} - \sum_{k=1}^{i-1} (d_{ki} d_{kj})}{a_{ii}} \qquad i \leq j \tag{295}$$

noting that

$$a_{ii} = S_{ij} - \sum_{k=1}^{i-1} (d_{ki})^2 \tag{296}$$

Let $D = \varphi B$, such that

$$D = \begin{bmatrix} \varphi_{11} & & & \\ & \varphi_{22} & & \\ & & \ddots & \\ & & & \varphi_{nn} \end{bmatrix} \begin{bmatrix} 1 & \frac{d_{12}}{\varphi_{11}} & \frac{d_{13}}{\varphi_{11}} & \cdots & \frac{d_{1n}}{\varphi_{11}} \\ & 1 & \frac{d_{23}}{\varphi_{22}} & & \frac{d_{2n}}{\varphi_{22}} \\ & & & & \\ & & & & 1 \end{bmatrix} \qquad (297)$$

The elements of $D^{-1} = d_{ij}^{-1}$ are

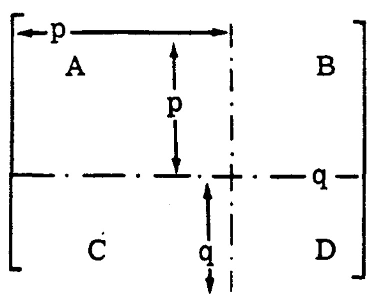
$$d_{ij}^{-1} = 0 \text{ for } i \geq j \qquad (298)$$

$$d_{ii}^{-1} = \frac{1}{\varphi_{ii}} = \frac{1}{d_{ii}} \qquad (299)$$

$$d_{ij}^{-1} = \sum_{k=i+1}^{n} b_{ik}\, d_{kj}^{-1} \text{ for } j \geq k \qquad (300)$$

4.0 PARTITIONING METHOD

Let the square matrix S, ($\neq 0$), be partitioned according to the following scheme:

S = 

$(p + q = n,\ p \geq q)$

and let the inverse S^{-1} be partitioned correspondingly according to

$$S^{-1} = \begin{bmatrix} K & L \\ M & N \end{bmatrix}$$

Since $S.S^{-1} = I$ the following relations are evident:

Set I

$$\left.\begin{aligned} N &= (D-CA^{-1}B)^{-1} \\ M &= (-N\,C\,A^{-1}) \\ L &= (-A^{-1}\,B\,N) \\ K &= (A^{-1}-A^{-1}\,B\,M) \end{aligned}\right\} \qquad (301)$$

Set II

$$\left.\begin{aligned} K &= (A-BD^{-1}C)^{-1} \\ L &= (K\,B\,D^{-1}) \\ M &= (D^{-1}\,C\,K) \\ N &= D^{-1}-D^{-1}\,CL \end{aligned}\right\} \qquad (302)$$

The solution involves the inversion of one $p \times p$ matrix and one $q \times q$ matrix; various matrix multiplications.

5.0 BORDERING TECHNIQUE

This is based on the partitioning of a $n \times n$ matrix A_n into a $(n-1) \times (n-1)$ matrix A_{n-1} whose inverse is known, a row matrix r, a column matrix c, and a single element, a_{nn}, according to the following scheme.

$$A_n = \left[\begin{array}{c|c} & a_{11} \\ A_{n-1} & \vdots \\ & a_{n-1.n} \\ \hline a_{n.1} \cdots a_{n.n-1} & a_{nn} \end{array}\right] = \left[\begin{array}{c|c} A_{n-1} & c_n \\ \hline r_n & a_{nn} \end{array}\right] \qquad (303)$$

The correspondingly partitioned inverse is

$$A_n^{-1} = \left[\begin{array}{c|c} AP_{n-1} & S_n \\ \hline q_n & 1/\alpha_n \end{array}\right] \qquad (304)$$

Since

$$A\,A^{-1} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$S_n = \frac{-A^{-1}_{n-1}\, c_n}{\alpha_n} \tag{305}$$

$$\alpha_n = a_{nn} - r_n\, A^{-1}_{n-1}\, c_n \tag{306}$$

$$P_{n-1} = A^{-1}_{n-1} - A^{-1}_{n-1}\, c_n\, q_n \tag{307}$$

$$q_n = \frac{-r_n\, A^{-1}_{n-1}}{\alpha_n} \tag{308}$$

so that

$$A^{-1}_n = \begin{bmatrix} A^{-1}_{n-1} + \dfrac{A^{-1}_{n-1}\, c_n\, r_n\, A^{-1}_{n-1}}{\alpha_n} \;, & -\dfrac{A^{-1}_{n-1}\, c_n}{\alpha_n} \\ -\dfrac{r_n\, A^{-1}_{n-1}}{\alpha_n} \;, & \dfrac{1}{\alpha_n} \end{bmatrix} \tag{309}$$

The inverse of the matrix is thus obtained by successive borderings, through the successive inversions of the sequence:

$$\left[a_{11}\right], \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ldots$$

Each step of this determination is carried out according to Equation (304).

In the event that the matrix to be inverted is symmetric, the bordering technique becomes much simpler.

Since the i^{th} row of the matrix is identical with the i^{th} column, the corrective terms

$$\frac{A_{n-1}^{-1} c_{n-1} r_{n-1} A_{n-1}^{-1}}{\alpha_n}$$

which are applied to A_{n-1}^{-1} may be determined element by element as

$$\Delta_{ij} = \frac{1}{\alpha_n} \prod_{\ell=1}^{n-1} \left[a_{i\ell} \; c \; c^r \; a_{j\ell} \right]$$

where Π denotes a product over the range of ℓ from the rows of the matrix A_{n-1}^{-1}.

Similarly α_n is determined according to $\alpha_n = a_{nn} - c_{n-1}^r A_{n-1}^{-1} c_{n-1}$, which is obtained from the rows of the matrix A_{n-1}^{-1}.

6.0 COMPARISON OF METHODS

Since the matrix which will be inverted is symmetric the preceding techniques will be somewhat simplified.

6.1 Single Division

It appears as if the single division scheme for $S = D^T D$ will yield the highest accuracy, since this method tends to make the leading diagonal of D closer to unity.

For this method, the storage for $n(n-1)/2$ elements must be provided, since the determination of the elements of the K^{th} row of D^{-1} utilizes all the elements of the rows $\ell \geq k$ which have already been determined. This disadvantage may be offset by suitable partitioning, but appears to become cumbersome.

Furthermore, the required inverse A^{-1} is not obtained until D^{-1} is post-multiplied by its transpose.

6.2 Partitioning

This has the advantage that it enables a matrix of any size to be inverted, provided that suitable partitioning has been performed. However, it is apparent that this becomes rather cumbersome, and is not recommended unless a large number of zero or unit partitions exist.

6.3 Bordering

It has been shown that the bordering technique for a schematic matrix may be obtained by the successive evaluation of algebraic expressions. These yield the elements of the inverse row by row, and computations are performed using rows of the previously determined inverse one at a time. Provided that these rows are output onto tape (and read back as required) a maximum of $(3n + 2)$ storage locations is required for the computations.

However, the method is more time consuming than the single division scheme.

6.4 Conclusion

It appears that the bordering technique is the most appropriate method to use, on account of the simpler programming and storage requirements.

The determination of $A A^{-1}$ and correcting of A^{-1} by Hotelling's method will probably have to be accomplished through a partitioning procedure. However, it is felt that it should be necessary to make only one iteration to obtain A^{-1} with sufficient accuracy.

In the event that a large number of iterations is necessary, the convergence may be improved by the methods described in Faddeeva Ch. 30.